COROLLARY 2. *Suppose $f_{-n}(x)$ is the associated factor sequence for $f(D)$ and $g_{-n}(x)$ is the associated factor sequence for $g(D)$, and suppose*

$$g_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} f_{-k}(x),$$

for constants $c_{-n,k}$. Then the sequence $r_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} x^{-k}$ is the associated factor sequence for $g(f^{-1}(D))$.

11. APPLICATIONS TO FORMAL POWER SERIES

Given a formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

we can define a linear functional L in P^* by $\langle L \mid x^k \rangle = a_k$. We call L the *generating functional* of the sequence a_k. The series $f(t)$ is the indicator of the linear functional L and $L = f(A)$.

When $a_0 = 0$ and $a_1 \neq 0$ we call $f(t)$ a *delta series*. We have seen that the composition $f(g(t))$ is well defined when the constant coefficient of $g(t)$ vanishes, in particular when $g(t)$ is a delta series, and that

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle f(g(A)) \mid x^k \rangle}{k!} t^k. \qquad (*)$$

If $f(t)$ is the indicator of the delta functional L, we have seen (Corollary 1 to Theorem 6) that

$$f^{-1}(t) = \sum_{k=0}^{\infty} \frac{\langle \tilde{L} \mid x^k \rangle}{k!} t^k.$$

That is, the reciprocal series $f^{-1}(t)$ is the indicator of $\tilde{L}$, the reciprocal functional to L.

If $f(t)$ and $g(t)$ are the indicators of the delta functionals L and M, then Theorem 6 tells us that $f(g(t))$ is the indicator of the delta functional $f(g(A)) = M \circ L$, and $(*)$ becomes

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle M \circ L \mid x^k \rangle}{k!} t^k. \qquad (**)$$

The problem of determining the composition of formal power series is thus equivalent to the problem of determining the composition of delta functionals. It turns out that the latter can often be explicitly computed by the present methods, as we shall see.

We can relate the composition of delta functionals to the umbral composition of sequences of binomial type. Suppose $p_n(x)$ and $q_n(x)$ are the conjugate sequences for the delta functionals $L = f(A)$ and $M = g(A)$, hence the associated sequences for $\tilde{L}$ and $\tilde{M}$. Then $q_n(\mathbf{p}(x))$ is the associated sequence for $\widetilde{\tilde{L} \circ \tilde{M}}$, and thus the conjugate sequence for $\widetilde{\tilde{L} \circ \tilde{M}} = (g^{-1} \circ f^{-1})^{-1}(A) = (f \circ g)(A) = M \circ L$. By definition therefore,

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} \frac{\langle (M \circ L)^k \mid x^n \rangle}{k!} x^k. \qquad (***)$$

Comparing $(**)$ and $(***)$, we see that the coefficient of $t^n/n!$ in $f(g(t))$ is the linear coefficient in $q_n(\mathbf{p}(x))$.

By definition of umbral composition, we have

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L^k \mid x^j \rangle \, x^k$$

and so $(***)$ gives

$$\langle (M \circ L)^k \mid x^n \rangle = \sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L^k \mid x^j \rangle.$$

Thus the coefficient of $t^n/n!$ in $f(g(t))$ is

$$\sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L \mid x^j \rangle. \qquad (****)$$

As an example, we compute the power series $(1 + g(t))^r - 1$, where r is a real number, and $g(t)$ is a delta series. Here $f(t)$ is the delta series $f(t) = (1 + t)^r - 1$. Expanding $L = f(A)$ in powers of A by means of the binomial series, we find that $\langle L \mid x^j \rangle = (r)_j - \delta_{j,0}$ and thus the coefficient of $t^n/n!$ in $(1 + g(t))^r - 1$ is

$$\sum_{j=1}^{n} \binom{r}{j} \langle M^j \mid x^n \rangle. \qquad (*****)$$

Formula $(****)$ yields at once Faà di Bruno's formula for the composition of two formal power series. The special cases of this formula to be found in the literature are obtained by explicitly computing a sequence of binomial type. For example, setting $g(t) = \log(1 + t)$, we find immediately from $(****)$ that the coefficients of $f(\log(1 + t))$ are given by umbral composition of $\phi_n(\mathbf{a})$, when $\phi_n(x)$ are the Stirling polynomials, and $\mathbf{a}$ is the umbral sequence a_n of coefficients of $f(t)$. Similarly, the coefficients of $f(e^t - 1)$ are given by $\phi_n(\mathbf{a})$, when $\phi_n(x)$ are the exponential polynomials.

8.2. We now compute the reciprocal polynomials to the Bell polynomials; that is, the associated sequence $p_n(x)$ for the delta functional $L = AN = x_1A + x_2A^2/2! + x_3A_3/3! + \cdots$. We may take $x_1 = 1$. We wish to use the Transfer Formula $p_n(x) = xP^{-n}x^{n-1}$, where $P = \mu(N)$, so we compute P^{-n}.

The indicator of P^{-n} is $(1 + g(t))^{-n}$, where $g(t)$ is the indicator of $M = N - \epsilon$. Hence, the coefficient of $D^k/k!$ in the expansion of P^{-n} is given by (*****), with $r = -n$ and $n = k$. The computation of (*****) is straightforward by binomial expansion and by the identity

$$\langle N^i \mid x^k \rangle = \frac{\langle L^i \mid x^{k+i} \rangle}{(k+i)_i} = \frac{i!\,k!}{(k+i)!} B_{k+i,i}\,.$$

We obtain

$$P^{-n} = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^i \frac{(n+j-1)_j}{(j-i)!\,(k+i)!} B_{k+i,i} D^k$$

and then

$$b_n(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^i \frac{(n+j-1)_{k+j}}{(j-i)!\,(k+i)!} B_{k+i,i} x^{n-k}.$$

Similarly, the associated factor sequence to the Bell polynomials is

$$f_{-n}(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^{k+j-i} \frac{(n+k)_{k+j}}{(j-i)!\,(k+i)!} B_{k+i,i} x^{-n-k}.$$

Umbral techniques can be used in several ways to compute power series expansions.

Consider the function $[\log(1+t)]^r$. If we take $f(t) = t^r + t$ and $g(t) = \log(1+t)$, then both $f(t)$ and $g(t)$ are delta series, and the expansion of $f(g(t))$ differs from the desired one only by the addition of $\log(1+t)$. To find the coefficients of $f(g(t))$ we compute the umbral composition $q_n(\mathbf{p}(x))$, where $q_n(x)$ is the conjugate sequence for $g(A)$ and $p_n(x)$ is the conjugate sequence for $f(A)$. The sequence $q_n(x)$ is the associated sequence for $g^{-1}(A) = e^A - \epsilon = \epsilon_1 - \epsilon$, and we have seen that $q_n(x) = (x)_n = \sum_{k=0}^{n} s(n,k)x^k$, where $s(n,k)$ are Stirling numbers of the first kind. It is even easier to compute the polynomials $p_n(x)$. For

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} \frac{\langle (A^r + A)^k \mid x^n \rangle}{k!} x^k \\ &= \sum_{k=0}^{n} \binom{k}{(n-k)/(n-1)} \frac{n!}{k!} x^k. \end{aligned}$$

Thus the linear coefficient in $q_n(\mathbf{p}(x))$ is $r!\,s(n, r) + s(n, 1) = r!\,s(n, r) + (-1)^{n-1}(n-1)!$, and so

$$[\log(1+t)]^r = \sum_{k=1}^{\infty} \frac{r!\,s(k, r)}{k!} x^k.$$

Consider next the function $\log(1 + \sin t)$. We have $f(t) = \log(1 + t)$ and $g(t) = \sin t$. Since $M = g(A) = \sin A$, by expansion we have

$$M^j = \frac{1}{(2i)^j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} e^{iA(2k-j)}$$

But $\langle e^{mA} \mid x^n \rangle = \langle (mA)^n/n! \mid x^n \rangle = m^n$ and so

$$\langle M^j \mid x^n \rangle = \frac{1}{(2i)^j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} i^n (2k-j)^n.$$

Now if $L = f(A) = \log(1 + A)$, then

$$\langle L \mid x^j \rangle = \langle [(-1)^{j+1}/j]\, A^j \mid x^j \rangle = (-1)^{j+1}(j-1)!$$

Thus the coefficient of $t^n/n!$ in $\log(1 + \sin t)$ is

$$\sum_{j=0}^{n} \sum_{k=0}^{j} \frac{(-1)^{k+1}}{j2^j} i^{n-j} \binom{j}{k} (2k-j)^n.$$

Next we give the generating functions of associated and Sheffer sequences. If $p_n(x)$ is the associated sequence for the delta functional $L = f(A)$, then by the Expansion Theorem

$$\epsilon_y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} f(A)^k.$$

Passing to indicators gives

$$e^{yt} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} f(t)^k.$$

Finally, replacing $f(t)$ by t gives

$$e^{yf^{-1}(t)} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k.$$

Thus, if $f(t)$ is a delta series its reciprocal is the series

$$f^{-1}(t) = \sum_{k=0}^{\infty} \frac{p_k'(0)}{k!} t^k,$$

where $p_n(x)$ are the associated polynomials to $f(A)$. Our recipe for finding the reciprocal of a formal power series is thus the following: Compute the associated polynomials, possibly by using the Recurrence Formula or the Transfer Formula, and then take the coefficients of x in these polynomials. It turns out that computing the whole polynomial sequence is often speedier than computing a single coefficient.

We turn to some more examples.

2.6. The exponential polynomials $\phi_n(x) = \sum_{k=0}^{n} S(n, k)x^k$ are the associated polynomials for $f^{-1}(A) = \log(1 + A)$. Thus

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} \frac{\phi_k(x)}{k!} t^k.$$

4.5. The Abel polynomials $p_n(x) = x(x - an)^{n-1}$ have the generating function

$$e^{xf^{-1}(t)} = \sum_{k=0}^{\infty} \frac{x(x - ak)^{k-1}}{k!} t^k,$$

where $f(t) = te^{at}$.

7.5. The basic Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-x)^k$$

have the generating function

$$e^{xt/(t-1)} = \sum_{k=0}^{\infty} \frac{L_k(x)}{k!} t^k.$$

If $s_n(x)$ is the Sheffer sequence for $N = f(t)$ with respect to the delta functional $L = g(t)$, then by Theorem 10

$$\epsilon_y N^{-1} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} L^k.$$

Taking indicators and simplifying as before, we find the generating function

$$\frac{1}{f(g^{-1}(t))} e^{yg^{-1}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k.$$

For the higher Laguerre polynomials $L_n^{(\alpha)}(x)$ we have $f(t) = (1 - t)^{-\alpha-1}$ and $g(t) = t(t - 1)^{-1}$, hence

$$(1 - t)^{-\alpha-1} e^{xt/(t-1)} = \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(x)}{k!} t^k.$$

For the Hermite polynomials of variance v, $H_n^{(v)}(x)$, we have $f(t) = e^{vt^2/2}$ and $g(t) = t$. Thus

$$e^{-vt^2/2}e^{xt} = \sum_{k=0}^{\infty} \frac{H_k^{(v)}(x)}{k!} t^k.$$

Lagrange's inversion formula is immediate in the present notation. It states that if $f(t)$ is a delta series, then the nth coefficient in $f^{-1}(t)^k$ equals the $(n-k)$th coefficient in $(f(t)/t)^{-n}$, multiplied by k/n. In our notation, this reads

$$\frac{\langle \tilde{L}^k \mid x^n \rangle}{n!} = \frac{k}{n} \frac{\langle M^{-n} \mid x^{n-k} \rangle}{(n-k)!},$$

where the indicator of $L = AM$ is $f(t)$. The verification of this fact is now a trivial computation with adjoints. If $p_n(x)$ is the associated sequence for L, then using the Transfer Formula we find

$$\begin{aligned} \langle \tilde{L}^k \mid x^n \rangle &= \langle A^k \mid p_n(x) \rangle = \langle A^k \mid x\mu(M)^{-n}\, x^{n-1} \rangle \\ &= \langle kA^{k-1} \mid \mu(M)^{-n}\, x^{n-1} \rangle = k\langle A^{k-1}M^{-n} \mid x^{n-1} \rangle \\ &= k\langle M^{-n} \mid D^{k-1}x^{n-1} \rangle = \frac{k}{n}\,(n)_k \,\langle M^{-n} \mid x^{n-k} \rangle, \end{aligned}$$

as desired.

We can just as easily prove the variants of the Lagrange inversion formulas, for example: Given two delta series $f(t)$ and $g(t)$, the nth coefficient in $g(f^{-1}(t))$, multiplied by n, equals the $(n-1)$st coefficient in $g'(t)(f(t)/t)^{-n}$. In symbols:

$$\langle g(f^{-1}(A)) \mid x^n \rangle = \left\langle g'(A) \left(\frac{f(A)}{A}\right)^{-n} \middle| \, x^{n-1} \right\rangle.$$

But this is also an immediate consequence of adjointness. Indeed, the right side can be written as

$$\langle g(A) \mid xP^{-n}x^{n-1} \rangle,$$

where $P = \mu(f(A)/A)$. We recognize an instance of the Transfer Formula: $\langle g(A) \mid p_n(x) \rangle$, where $p_n(x)$ are the associated polynomials for $f(A)$. Letting α be the umbral operator mapping x^n to $p_n(x)$, and recalling that the automorphism α^* maps $f(A)$ to A, we have

$$\begin{aligned} \langle g(A) \mid p_n(x) \rangle &= \langle g(A) \mid \alpha x^n \rangle \\ &= \langle \alpha^* g(A) \mid x^n \rangle = \langle g(f^{-1}(A)) \mid x^n \rangle. \end{aligned}$$

It is hard to imagine a simpler proof.

A variant of the same reasoning gives Hermite's version of the Lagrange inversion formula, namely,

$$\left\langle \frac{Ag(f^{-1}(A))}{f^{-1}(A)\,f'(f^{-1}(A))} \,\middle|\, x^n \right\rangle = \left\langle g(A) \left(\frac{f(A)}{A}\right)^{-n} \,\middle|\, x^n \right\rangle.$$

The generating functions of factor sequences cannot be expressed by ordinary generating functions, and lead us to introduce an analogous formal device. Let $f(t)$ be a delta series and let $g(t)$ be a formal power series. We define the *Cigler transform* of the pair (f, g), in symbols

$$F(x) = \int_{-\infty}^{0} g(t)\, e^{xf(t)}\, dt$$

to be the formal power series obtained after term-by-term integration of

$$\int_{-\infty}^{0} g(f^{-1}(s))(f^{-1})'\,(s)\, e^{sx}\, ds.$$

The point is that one can compute with the Cigler transform in much the same way as with an ordinary integral, for example,

$$\int_{-\infty}^{0} g(t)\, e^{xf(t)}\, dt + \int_{-\infty}^{0} g(t)\, e^{xh(t)}\, dt = \int_{-\infty}^{0} g(t)\, e^{x(f(t)+h(t))}\, dt;$$

thus the Cigler transform is an "integral" analog of a formal power series.

If $f_{-n}(x)$ is the associated factor sequence of the delta operator $Q = f(D)$, then

$$\begin{aligned} f_{-1}(x) &= Q' \frac{1}{x} = \int_{-\infty}^{0} f'(t)\, e^{xt}\, dt \\ &= \int_{-\infty}^{0} e^{xf^{-1}(t)}\, dt, \end{aligned}$$

and more generally, applying Q successively,

$$f_{-n}(x) = \frac{(-1)^{n-1}}{(n-1)!} \int_{-\infty}^{0} t^{n-1} e^{xf^{-1}(t)}\, dt;$$

thus the generating function of $f_{-n}(x)$ can be expressed by the Cigler transform:

$$\sum_{k \geqslant 1} f_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xf^{-1}(t)}\, dt.$$

Similarly, if $g_n(x) = Tf_{-n}(x)$ is the factor sequence obtained from the associated factor sequence by applying the invertible shift-invariant operator

$T = g(D)$, then by a Cigler transform one can express the generating function of $g_{-n}(x)$ in the form

$$g_{-1}(x) = TQ'\frac{1}{x} = \int_{-\infty}^{0} g(t)\, f'(t)\, e^{xt}\, dt$$
$$= \int_{-\infty}^{0} g(f^{-1}(t))\, e^{xf^{-1}(t)}\, dt,$$

whence

$$g_{-n}(x) = Q^{n-1}TQ'\frac{1}{x} = \int_{-\infty}^{0} \frac{(-1)^{n-1}\, t^{n-1}}{(n-1)!}\, g(f^{-1}(t))\, e^{xf^{-1}(t)}\, dt$$

and again

$$\sum_{k\geqslant 1} g_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} g(f^{-1}(t))\, e^{-st+xf^{-1}(t)}\, dt.$$

12. Examples of Factor Sequences

2.7. The *negative factorial powers*

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)}$$

are the associated factor sequence for the operator $\Delta = \sigma(\epsilon_1 - \epsilon)$. This follows from the Recurrence Formula:

$$\Delta' x^{-1}(x)_{-n} = E^1 x^{-1}(x)_{-n} = (x)_{-n-1}\,.$$

Thus we immediately have

$$(x+y)_{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (y)_k\, (x)_{-n-k}\,,$$

as well as

$$\Delta^m (x)_{-n} = (-n)_m (x)_{-n-m}\,.$$

Corollary 2 to Theorem 17 gives

$$(x)_{-n} = \sum_{k=0}^{\infty} (-1)^k\, \frac{\langle L^n \mid x^{n+k}\rangle}{n!}\, x^{-n-k},$$

where $L = \epsilon_1 - \epsilon$. But $q_n(x) = \sum_{k=0}^{n} (\langle L^k \mid x^n\rangle / k!)\, x^k$ is the conjugate sequence for L. Thus

$$\langle L^n \mid x^{n+k}\rangle / n! = S(n+k, n),$$

where $S(n, k)$ are the Stirling numbers of the second kind. Therefore, we obtain the identity

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=0}^{\infty} (-1)^k\, S(n+k, n)\, x^{-n-k}.$$

3.5. A similar treatment may be given to the associated factor sequence for $\nabla = \sigma(\epsilon - \epsilon_{-1})$, which is

$$\langle x\rangle_{-n} = \frac{(-1)^n}{(x-1)(x-2)\cdots(x-n)}.$$

We state as a sample:

$$\langle x\rangle_{-n} = \sum_{k=0}^{\infty} S(n+k, n)\, x^{-n-k}.$$

4.6. The associated factor sequence for the Abel operator De^{aD} is given most easily by the Transfer Formula:

$$\begin{aligned} A_{-n}(x, a) &= xE^{an}x^{-n-1} \\ &= x(x+an)^{-n-1}. \end{aligned}$$

Thus the identity

$$(x+y)(x+y+an)^{-n-1} = \sum_{k=0}^{\infty} \binom{-n}{k} xy(y+ak)^{k-1}\,(x+a(n-k))^{-n-k-1}$$

is immediate.

Corollary 2 of Theorem 17 yields

$$A_{-n}(x, a) = \sum_{k=0}^{\infty} \binom{-n-1}{k} (an)^k\, x^{-n-k}.$$

6.6. The *negative Steffensen polynomials* are the associated factor sequence for $e^{-D/2}(e^D - 1)$, and thus by Corollary 1 to Theorem 17,

$$\begin{aligned} x^{[-n]} &= xE^{-n/2}x^{-1}(x)_{-n} \\ &= x(x - n/2 - 1)_{-n-1} \\ &= \frac{x}{(x-n/2)(x-n/2+1)\cdots(x+n/2)}. \end{aligned}$$

7.6. The associated factor sequence for the Laguerre operator $D/(D-I)$ is

$$\begin{aligned} L_{-n}(x) &= x(D-I)^{-n}\, x^{-n-1} \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^{-n-k}\, (-n-1)_k\, x^{-n-k}, \end{aligned}$$

this by the Transfer Formula.

In view of Theorem 18, the factor sequence $L_{-n}(x)$ is self-reciprocal, as expected.

If $f_{-n}(x)$ is the associated factor sequence for the delta functional L, then the *conjugate factor sequence* $g_{-n}(x)$ to $f_{-n}(x)$ is the associated factor sequence for $\tilde{L}$, the reciprocal to L. By Theorem 18, we have $f_{-n}(g(x)) = g_{-n}(\mathbf{f}(x)) = x^{-n}$.

2.8. The *negative exponential polynomials* $\phi_{-n}(x)$ are the conjugate factor sequence to $(x)_{-n}$, and are therefore the associated factor polynomials for $\log(I + D)$.

Corollary 2 to Theorem 17 gives

$$\phi_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k s(n+k, n)\, x^{-n-k},$$

where $s(n, k)$ are the Stirling numbers of the first kind.

We have by Theorem 18 the umbral substitutions,

$$\begin{aligned} x^{-n} &= \sum_{k=0}^{\infty} (-1)^k s(n+k, n)(x)_{-n-k} \\ &= \sum_{k=0}^{\infty} (-1)^k S(n+k, n)\, \phi_{-n-k}(x), \end{aligned}$$

which are equivalent to the Stirling number identities

$$\sum_{k=0}^{j} s(n+k, n)\, S(n+j, n+k) = \delta_{j,0}$$

and

$$\sum_{k=0}^{j} S(n+k, n)\, s(n+j, n+k) = \delta_{j,0}\,.$$

By the Recurrence Formula,

$$\phi_{-n-1}(x) = (I+D)^{-1}\, x^{-1} \phi_{-n}(x)$$

and so

$$\begin{aligned} \phi_{-n}(x) &= x(I+D)\, \phi_{-n-1}(x) \\ &= \cdots = [x(I+D)]^k\, \phi_{-n-k}(x). \end{aligned}$$

Taking $n = 1$ and $k = n$ gives

$$\begin{aligned} \phi_{-1}(x) &= [x(I+D)]^n\, \phi_{-1-n}(x) \\ &= e^{-x}(xD)^n\, e^x \phi_{-1-n}(x). \end{aligned}$$

3.6. The conjugate factor sequence to the sequence $A_{-n}(x, a)$ is computed by Corollary 2 to Theorem 17:

$$\mu_{-n}(x, a) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle \tilde{L}^n \mid x^{n+k} \rangle}{n!} x^{-n-k},$$

where $\tilde{L}$ is the reciprocal to $\epsilon_a A$. However,

$$\frac{\langle \tilde{L}^n \mid x^{n+k} \rangle}{n!} = \binom{n+k-1}{n-1} \frac{[-a(n+k)]^k}{n!}$$

and thus

$$\mu_{-n}(x, a) = \sum_{k=0}^{\infty} \binom{-n}{k} \frac{[-a(n+k)]^k}{n!} x^{-n-k}.$$

The umbral identity $x^{-n} = \mu_{-n}(\mathbf{A}(x, a), a)$ gives the elegant power series identity:

$$x^{-n-1} = \sum_{k=0}^{\infty} \binom{-n}{k} \frac{[-a(n+k)]^k}{n!} [x + a(n+k)]^{-n-k-1}.$$

We turn now to some connection-constant problems.

2.9. Determine the connection constants $c_{-n,k}$ in

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=1}^{\infty} \frac{c_{-n,k}(-1)^k}{(x-1)(x-2)\cdots(x-k)}.$$

Since $(x)_{-n}$ is the associated factor sequence for $g(D) = e^D - I$ and $\langle x \rangle_{-k}$ is the associated factor sequence for $f(D) = I - e^{-D}$, we have $g(f^{-1}(D)) = D/(I - D)$ and so

$$\sum_{k=1}^{\infty} c_{-n,k} x^{-k} = L_{-n}(-x).$$

Thus

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=1}^{\infty} \binom{-n}{k-n} \frac{(-n-1)_{k-n}}{(x-1)(x-2)\cdots(x-k)}.$$

6.7. Determine the constants $c_{-n,k}$ in

$$\frac{x}{(x-n/2)(x-n/2+1)\cdots(x+n/2)} = \sum_{k=1}^{\infty} \frac{c_{-n,k}}{(x+1)(x+2)\cdots(x+k)}.$$

Since $x^{[-n]}$ is the associated factor sequence for $g(D) = e^{-D/2}(e^D - I)$ and

$(x)_{-n}$ is the associated factor sequence for $f(D) = e^D - I$, we have $g(f^{-1}(D)) = D(I + D)^{-1/2}$. Therefore,

$$\sum_{k=1}^{\infty} c_{-n,k} x^{-k} = x(I + D)^{-n/2}\, x^{-n-1}$$

$$= \sum_{k=1}^{\infty} \binom{-n/2}{k} (-n-1)_k\, x^{-n-k},$$

and so

$$c_{-n,k} = \binom{-n/2}{k-n} (-n-1)_{k-n}\,.$$

7.8. Determine the constants $c_{-n,k}$ relating the Laguerre polynomials to the exponential polynomials:

$$L_{-n}(x) = \sum_{k=1}^{\infty} c_{-n,k} \phi_{-k}(-x).$$

Since $L_{-n}(x)$ is the associated factor sequence for $g(D) = D/(D - I)$ and $\phi_{-n}(-x)$ is the associated factor sequence for $f(D) = \log(I - D)$, we have $g(f^{-1}(D)) = I - e^{-D}$ and so

$$\sum_{k=1}^{\infty} c_{-n,k} x^{-k} = \langle x \rangle_{-n}\,.$$

Thus

$$c_{-n,k} = S(k, n)$$

and

$$L_{-n}(x) = \sum_{k=1}^{\infty} S(k, n)\, \phi_{-k}(-x).$$

We postpone discussion of Hermite and higher-order Laguerre factor sequences until Section 13.

We conclude with some examples of Cigler transforms.

2.10. For the factor sequence $(x)_{-n}$, we obtain as a special case of the Cigler transform Nielsen's factorial expansion of the incomplete gamma function:

$$\sum_{k \geqslant 1} (x)_{-k}\, s^{k-1} = \int_{-\infty}^{0} e^{-st}(1 + t)^x\, dt.$$

4.9. For the Abel sequence $A_{-n}(x, a)$ we obtain

$$\sum_{k\geqslant 1} A_{-k}(x, a)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xf^{-1}(t)}\, dt$$
$$= \int_{-\infty}^{0} e^{-st} \left(\frac{t}{f^{-1}(t)}\right)^{x/a} dt,$$

where

$$f(t) = te^{at}.$$

7.9. For the Laguerre sequence $L_{-n}(x)$ we obtain

$$\sum_{k\geqslant 1} L_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xt/(t-1)}\, dt.$$

2.11. For the negative exponential polynomials:

$$\sum_{k\geqslant 1} \phi_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+x(e^t-1)}\, dt.$$

4.10. For the sequence $\mu_{-n}(x, a)$, reciprocal to the Abel factor sequence:

$$\sum_{k\geqslant 1} \mu_{-k}(x, a)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xte^{at}}\, dt.$$

13. Hermite and Laguerre Polynomials

Theories of special functions often present those functions that are of frequent occurrence as special cases of some general concept, and the present development is no exception. In actual fact, however, those special sequences of polynomials that have actually occurred are best defined by their own structural conditions. Such axiomatic descriptions remain largely undiscovered, partially because of a deficiency of notational suppleness in the theory of special functions which it is the avowed purpose of the present work to remedy.

Hermite Polynomials

As an instance of such a structural characterization, we consider the following problem: Find all Appell sequences $s_n(x)$ with the property that

$$\langle s_j(A) \mid s_n(x)\rangle = -(1/v)\langle s_{j-1}(A) \mid s_{n+1}(x)\rangle,$$

for some constant v.

By the Recurrence Formula,

$$s_{n+1}(x) = (x + T(T^{-1})')\, s_n(x)$$
$$= (x + S)\, s_n(x),$$

where $s_n(x) = T^{-1}x^n$. But this gives $s_j(A) = -(1/v)(\partial_A + S^*)\, s_{j-1}(A)$, whence S^* is multiplication by $-vA$ and $S = -vD$, and thus $T = e^{-vD^2/2}$. The resulting polynomials are the Hermite polynomials $H_n^{(v)}(x) = e^{-vD^2/2}x^n$ of variance v. For $v = 1$, we obtain the classical Hermite polynomials.

The elementary properties of the Hermite polynomials have been derived in "Finite Operator Calculus." We shall give a sampling of applications of the present methods. From the operational formula

$$(x - vD)\, p(x) = -e^{x^2/2v}(vD)\, e^{-x^2/2v}p(x),$$

one infers the recurrence

$$H_{n+j}^{(v)}(x) = (x - vD)^j\, H_n(x) = (-1)^j\, e^{x^2/2v}(vD)^j\, e^{-x^2/2v}H_n^{(v)}(x), \qquad (*)$$

and for $n = 0$ the Rodrigues formula

$$H_n^{(v)}(x) = (-1)^n\, e^{x^2/2v}(vD)^n\, e^{-x^2/2v}.$$

Expanding $(*)$ by the Leibnitz formula gives

$$H_{n+j}^{(v)}(x) = \sum_{k=0}^{j} (-v)^{j-k} \binom{j}{k} (n)_{j-k}\, H_k^{(v)}(x)\, H_{n-j+k}^{(v)}(x). \qquad (**)$$

Replacing n by $n - 2m$ and setting $j = n$, Eq. $(**)$ becomes

$$H_{2n-2m}^{(v)}(x) = \sum_{k=0}^{n} (-v)^{n-k} \binom{n}{k} (n - 2m)_{n-k}\, H_k^{(v)}(x)\, H_{k-2m}^{(v)}(x).$$

We recognize an umbral composition with the Laguerre polynomials

$$L_n^{(-2m)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - 2m)_{n-k}\, x^k.$$

Using the fact that the Laguerre polynomials are self-reciprocal (proved later in this section, or see "Finite Operator Calculus"), we obtain

$$v^{2n}H_n^{(v)}(x)\, H_{n-2m}^{(v)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n - 2m)_{n-k}\, H_{2k-2m}^{(v)}(x).$$

Changing n to $n+m$ gives

$$v^{2n+2m}H_{n+m}^{(v)}(x)\,H_{n-m}^{(v)}(x)=\sum_{k=0}^{n+m}\binom{n+m}{k}(n-m)_{n+m-k}\,H_{2k-2m}^{(v)}(x),$$

and finally letting $j=k-m$ we have

$$v^{2n+2m}H_{n+m}^{(v)}(x)\,H_{n-m}^{(v)}(x)=\sum_{j=m}^{n}\binom{n+m}{j+m}(n-m)_{n-j}\,H_{2j}^{(v)}(x).$$

From this formula we obtain

$$\begin{aligned}&v^{2n}[H_n^{(v)}(x)]^2-v^{n+m}H_{n-m}^{(v)}(x)\,H_{n+m}^{(v)}(x)\\&\quad=\sum_{j=0}^{n}\frac{(n-m)!}{(j-m)!}\binom{n}{j}\left[\binom{n}{m}\binom{j+m}{m}-\binom{j}{m}\binom{n+m}{m}\right]\Big/\binom{j}{m}\binom{j+m}{m}\,H_{2j}^{(v)}(x).\end{aligned}$$

Now for negative variance v, we know (Finite Operator Calculus) that $H_{2j}^{(v)}(x)$ is nonnegative, and since the above coefficients are nonnegative, we obtain

$$v^{2n}[H_n^{(v)}(x)]^2-v^{n+m}H_{n-m}^{(v)}(x)\,H_{n+m}^{(v)}(x)\geqslant 0.$$

For m even, we obtain a Turán-type inequality.

We give now a duplication formula for Hermite polynomials. That is, we determine the connection constants $c_{n,k}$ in

$$H_n^{(v)}(ax)=\sum_{k=0}^{n}c_{n,k}H_k^{(w)}(x).$$

Since $H_n^{(w)}(x)$ is the Sheffer sequence for the pair $(e^{wA^2/2}, A)$ and $H_n^{(v)}(ax)$ is Sheffer for the pair $(e^{va^{-2}A^2/2}, a^{-1}A)$, we have by Corollary 2 to Proposition 9.5 that $t_n(x)=\sum_{k=0}^{n}c_{n,k}x^k$ is Sheffer for the pair $(e^{(va^{-2}-w)A^2/2}, a^{-1}A)$ and so

$$\begin{aligned}t_n(x)&=a^ne^{(w-va^{-2})D^2/2}x^n\\&=\sum_{k=0}^{n}a^n\frac{(w-va^{-2})^k}{2^k}(n)_{2k}\,x^{n-2k}.\end{aligned}$$

We next determine the connection constants $c_{n,k}$ connecting the Hermite polynomials to the Bernoulli polynomials:

$$H_n^{(v)}(x)=\sum_{k=0}^{n}c_{n,k}B_k^{(\alpha)}(x).$$

Since $H_n^{(v)}(x)$ is Sheffer for the pair $(e^{vA^2/2}, A)$ and $B_n^{(\alpha)}(x)$ is Sheffer for the pair

$(((e^A - \epsilon)/A)^\alpha, A)$, the sequence $t_n(x) = \sum_{k=0}^n c_{n,k}x^k$ is Sheffer for the pair $(((e^A - \epsilon)/A)^{-\alpha} e^{vA^2/2}, A)$ and thus

$$t_n(x) = e^{-vD^2/2} \left(\frac{e^D - I}{D}\right)^\alpha x^n.$$

The constants $c_{n,k}$ are then determined by a routine Taylor's expansion.

Laguerre Polynomials

We have seen that the basic Laguerre polynomials arise in computing the connection constants between $(x)_n$ and $\langle x\rangle_n$. We now consider the more general problem of computing the connection constants between $E^{-\alpha-1}(x)_n$ and $\langle x\rangle_n$, for α a real number. More explicitly, we determine the constants $c_{n,k}$ in

$$(x - \alpha - 1)(x - \alpha - 2) \cdots (x - \alpha - n) = \sum_{k=0}^n c_{n,k}x(x + 1) \cdots (x + n - 1).$$

The sequence $E^{-\alpha-1}(x)_n$ is Sheffer for the pair $(e^{(\alpha+1)A}, e^A - \epsilon)$ and the sequence $\langle x\rangle_n$ is Sheffer for the pair $(\epsilon, \epsilon - e^{-A})$. Thus Corollary 2 to Proposition 9.5 tells us that $t_n(x) = \sum_{k=0}^n c_{n,k}x^k$ is Sheffer for the pair $((\epsilon - A)^{-\alpha-1}, A/(\epsilon - A))$. Thus

$$t_n(x) = (I - D)^{\alpha+1} L_n(-x).$$

The Laguerre polynomials of order α are

$$L_n^{(\alpha)}(x) = (I - D)^{\alpha+1} L_n(x).$$

The ubiquitous presence of these polynomials can be traced to the fact that they give this important set of connection constants. The reader is referred to "Finite Operator Calculus" for the elementary properties of the Laguerre polynomials. We cite only

$$L_n^{(\alpha+\beta+1)}(x + y) = \sum_{k\geqslant 0} \binom{n}{k} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y) \qquad (*)$$

and the formula due to Kahaner, Odlyzko, and Rota:

$$L_n^{(\alpha_1)}(\mathbf{L}^{(\alpha_2)}(\mathbf{L}^{(\alpha_3)}(\cdots \mathbf{L}^{(\alpha_k)}(x) \cdots))) = \begin{cases} L_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), & k \text{ odd}, \\ M_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), & k \text{ even}, \end{cases} \qquad (**)$$

where $M_n^{(\lambda)}(x) = (I - D)^{-\lambda} x^n$.

Equation $(*)$ gives the connection constants between Laguerre polynomials of different orders. Equation $(**)$, for $k = 2$, gives $L_n^{(\alpha_1)}(\mathbf{L}^{(\alpha_2)}(x)) = (I - D)^{\alpha_2-\alpha_1} x^n = (-1)^n L_n^{(\alpha_2-\alpha_1-n)}(x)$. For $\alpha_1 = \alpha_2 = \alpha$, we obtain $L_n^{(\alpha)}(\mathbf{L}^{(\alpha)}(x)) = x^n$ showing that all the Laguerre polynomials are self-reciprocal.

Various representations of the Laguerre polynomials of Rodrigues type follow from our methods. As an example, we prove Carlitz's beautiful:

$$L_n^{(\alpha)}(x) = \langle XD - X + \alpha + 1\rangle_n \, 1.$$

This formula is a consequence of Theorem 14. In the notation of the theorem, we have $(Q')^{-1} = -(D - I)^2$, whence the corresponding shift operator is $\theta = -X(D - I)^2$. Similarly, $P = (I - Q)^{\alpha+1}$, so that $P\partial_Q P^{-1} = (\alpha + 1)(I - D)$. Thus

$$\begin{aligned} L_n^{(\alpha)}(x) &= (-(\alpha + 1)(D - I) - X(D - I)^2)\, L_{n-1}^{(\alpha)}(x) \\ &= (XD - X + \alpha + 1)(I - D)\, L_{n-1}^{(\alpha)}(x) \\ &= (XD - X + \alpha + 1)\, L_{n-1}^{(\alpha+1)}(x) \\ &= \cdots = \langle XD - X + \alpha + 1\rangle_n \, 1. \end{aligned}$$

More generally, we have proved

$$L_{m+n}^{(\alpha)}(x) = \langle XD - X + \alpha + 1\rangle_n \, L_m^{(\alpha+m)}(x).$$

We derive the Erdelyi duplication formula for Laguerre polynomials:

$$L_n^{(\alpha)}(ax) = \sum_{k=0}^{n} c_{n,k} L_k^{(\alpha)}(x).$$

Since $L_n^{(\alpha)}(x)$ is Sheffer for the pair $((\epsilon - A)^{-\alpha-1}, A/(A - \epsilon))$ and $L_n^{(\alpha)}(ax)$ is Sheffer for the pair $((\epsilon - a^{-1}A)^{-\alpha-1}, A/(A - a))$ the sequence $t_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$ is Sheffer for the pair $(a^{\alpha+1}(a + (1 - a)A)^{-\alpha-1}, A/(a + (1 - a)A))$. By the Transfer Formula,

$$\begin{aligned} t_n(x) &= a^{-\alpha}(a + (1 - a)\, D)^{\alpha+n}\, x^n \\ &= \sum_{k=0}^{n} \binom{\alpha + n}{n - k} a^k (1 - a)^{n-k} (n)_{n-k}\, x^k. \end{aligned}$$

Factor Hermite Sequences

The *factor Hermite sequence* of variance v is the factor sequence for the pair $(e^{-vD^2/2}, D)$. Thus

$$H_{-n}^{(v)}(x) = e^{-vD^2/2} x^{-n}.$$

We have

$$H_{-n}^{(v)}(x + y) = \sum_{k=0}^{\infty} \binom{-n}{k} y^k H_{-n-k}^{(v)}(x).$$

Corollary 3 to Theorem 17 gives

$$\begin{aligned} H_{-n}^{(v)}(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{\langle e^{-vA^2/2} nA^{n-1} \mid x^{n+k-1} \rangle}{n!} x^{-n-k} \\ &= \sum_{k=0}^{\infty} \left(-\frac{v}{2}\right)^k (-n)_{2k}\, x^{-n-2k}. \end{aligned}$$

Theorem 19 easily establishes the umbral composition

$$H_{-n}^{(v)}(\mathbf{H}_{-}^{(w)}(x)) = H_{-n}^{(v+w)}(x).$$

The Cigler transform for $H_{-n}^{(v)}(x)$ gives

$$\sum_{k \geqslant 1} H_{-k}^{(v)}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-vt^2/2-(s-x)t}\, dt.$$

We establish the duplication formula for factor Hermite sequences

$$H_{-n}^{(v)}(ax) = \sum_{k=1}^{\infty} c_{-n,k} H_{-k}^{(w)}(x).$$

It is not hard to see that if $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence, then $f_{-n}(ax)$ is the $(f(a^{-1}D), g(a^{-1}D))$-factor sequence. Thus since $H_{-n}^{(v)}(ax)$ is the factor sequence for the pair $(e^{-va^{-2}D^2/2}, D)$ and $H_{-n}^{(w)}(x)$ is the factor sequence for the pair $(e^{-wD^2/2}, D)$, the sequence $r_{-n}(x) = \sum_{k=1}^{\infty} c_{-n,k} x^{-k}$ is the factor sequence for the pair $(e^{(w-va^{-2})D^2/2}, a^{-1}D)$ and thus

$$\begin{aligned} r_{-n}(x) &= a^{-n} e^{(w-va^{-2})D^2/2} x^{-n} \\ &= \sum_{k=0}^{\infty} a^{-n} \frac{(w-va^{-2})^k}{2^k} (-n)_{2k}\, x^{-n-2k}. \end{aligned}$$

Factor Laguerre Sequences

The *factor Laguerre sequence* of order α is the factor sequence for the pair $((I-D)^{\alpha+1}, D/(D-I))$. Thus, by analogy with $L_n^{(\alpha)}(x)$, we have

$$\begin{aligned} L_{-n}^{(\alpha)}(x) &= (I-D)^{\alpha+1} L_{-n}(x) \\ &= (-1)^n (I-D)^{\alpha-n} x^{-n} \\ &= \sum_{k=0}^{\infty} \binom{\alpha-n}{n-k} (-1)^k (-n)_{n-k}\, x^{-k}. \end{aligned}$$

We have

$$L_{-n}^{(\alpha)}(x+y) = \sum_{k=0}^{\infty} \binom{-n}{k} L_k(y)\, L_{-n-k}^{(\alpha)}(x),$$

and applying the operator $(I - D)^{\beta+1} = \sigma(\epsilon - A)^{\beta+1}$ we obtain the composition law

$$L_{-n}^{(\alpha+\beta+1)}(x + y) = \sum_{k=0}^{\infty} \binom{-n}{k} L_k^{(\beta)}(y)\, L_{-n-k}^{(\alpha)}(x).$$

Theorem 19 implies

$$\begin{aligned} L_{-n}^{(\alpha)}(\mathbf{L}_{-}^{(\beta)}(x)) &= (I - D)^{\alpha-\beta}\, x^{-n} \\ &= (-1)^n\, L_{-n}^{(\alpha-\beta+n)}(x). \end{aligned}$$

For $\alpha = \beta$, we obtain the identity

$$L_{-n}^{(\alpha)}(\mathbf{L}_{-}^{(\alpha)}(x)) = x^{-n},$$

showing that the factor Laguerre sequence $L_{-n}^{(\alpha)}(x)$ is self-reciprocal. Explicitly we have

$$x^{-n} = \sum_{k=0}^{\infty} \binom{\alpha - n}{n - k} (-1)^k\, (-n)_{n-k}\, L_{-k}^{(\alpha)}(x).$$

The Cigler transform for the sequence $L_{-n}^{(\alpha)}(x)$ gives

$$\sum_{k=1}^{\infty} L_{-k}^{(\alpha)}(x)\, s^{k-1} = \int_{-\infty}^{0} (1 - t)^{-\alpha-1}\, e^{-st+xt/(t-1)}\, dt.$$

We determine the connection constants $c_{-n,k}$ in

$$\frac{1}{(x - \alpha)(x - \alpha + 1) \cdots (x - \alpha - 1 + n)} = \sum_{k=0}^{\infty} \frac{(-1)^k\, c_{-n,k}}{(x - 1)(x - 2) \cdots (x - k)}\,.$$

Just as before, $E^{-\alpha-1}(x)_{-n}$ is the factor sequence for the pair $(e^{-(\alpha+1)D}, e^D - I)$ and $\langle x\rangle_n$ is the factor sequence for the pair $(I, I - e^{-D})$. Thus Corollary 1 to Theorem 19 implies that $r_{-n}(x) = \sum_{k=1}^{\infty} c_{n,k} x^{-k}$ is the factor sequence for the pair $((I - D)^{\alpha+1}, D/(I - D))$. Hence

$$\begin{aligned} r_{-n}(x) &= L_{-n}^{(\alpha)}(-x) \\ &= \sum_{k=1}^{\infty} \binom{\alpha - n}{n - k} (-n)_{n-k}\, x^{-k}, \end{aligned}$$

as expected.

Finally, we derive the duplication formula for factor Laguerre sequences. Namely, we determine the constants $c_{-n,k}$ in

$$L_{-n}^{(\alpha)}(ax) = \sum_{k=1}^{\infty} c_{-n,k} L_{-k}^{(\alpha)}(x).$$

Since $L_{-n}^{(\alpha)}(x)$ is the $((I-D)^{\alpha+1}, D/(D-I))$-factor sequence, Corollary 1 to Theorem 19 implies that $r_{-n}(x) = \sum_{k=1}^{\infty} c_{-n,k} x^{-k}$ is the factor sequence for the pair $(a^{-\alpha-1}(a+(1-a)D)^{\alpha+1}, D/(a+(1-a)D))$. Thus

$$\begin{aligned} r_{-n}(x) &= a^{-\alpha}(a+(1-a)\,D)^{\alpha-n}\,x^{-n} \\ &= \sum_{k=1}^{\infty} \binom{\alpha-n}{n-k} a^{-k}(1-a)^{n-k}\,(-n)_{n-k}\,x^{-k}. \end{aligned}$$

14. Applications to Combinatorics

We define a *store* σ as a set, in general infinite, together with a map d which assigns to every element of σ a positive integer, called its *degree*. The subset of σ consisting of all elements of a given degree is assumed to be finite. In practice, the elements of σ are sets endowed with some structure, and the problem is to count σ; that is, to determine the number a_n of elements of σ of degree n. We call a_n the *counting sequence* of σ, and we assume that $a_1 > 0$.

We define the *generating functional* of σ as the delta functional L satisfying

$$\langle L \mid x^n \rangle = a_n\,.$$

The counting sequence is thus the sequence of coefficients of the indicator of L.

The *partitional* of a store σ (a translation of Foata's "composé partitionnel") is a second store *part* (σ) defined as follows. An element p of part (σ) is a set (not a sequence) of pairs $\{(B_1, s_1),..., (B_k, s_k)\}$, where

(i) the B_i are the blocks of a partition of the set $\{1, 2,..., n\}$, for some n (hence B_i is nonempty);

(ii) the s_i are elements of the store σ;

(iii) the degree of s_i equals the number of elements in B_i.

To every such element p, called a *part* of part (σ), we associate two integers; the *degree* $d(p)$ of p is the sum of the degrees of s_i and the *part number* of p is the number of blocks.

The partitional part (σ) is obtained by letting n range over all positive integers. We let $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, and call it the *counting sequence* of the partitional. We set $b_{0,0} = 1$. Since $a_1 > 0$, we have $b_{n,n} > 0$ for all n.

The following proposition motivates this definition.

Proposition 14.1. *Let $b_{n,k}$ be the counting sequence of the partitional part (σ) of a store σ having generating functional L, for the degree n and the part number k. Then*

$$b_{n,k} = \langle L^k \mid x^n \rangle / k!.$$

Proof. Evidently the counting sequence satisfies the identity

$$\binom{i+j}{i} b_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} b_{k,i} b_{n-k,j}\,.$$

Therefore, by Proposition 4.3, there exists a delta functional M such that

$$b_{n,k} = \langle M^k \mid x^n \rangle / k!.$$

But it is immediate from the definition of partitional that $b_{n,1} = a_n$. Thus $M = L$. Q.E.D.

We remark that $\sum_{k=0}^{n} b_{n,k} x^k$ is the conjugate sequence for the delta functional L.

COROLLARY 1. *Let* $c_n = \sum_{k \geqslant 0} b_{n,k}$ *be the number of elements of degree* n *in part* (σ). *Then*

$$c_n = \langle e^L \mid x^n \rangle.$$

COROLLARY 2 (Foata). *The exponential generating function of* c_n *is the indicator of the exponential of the generating functional of* σ.

We illustrate these notions with some elementary examples.

EXAMPLE 1. Find the number of partitions of an n-set.

Solution. Let σ be the store having exactly one element of each degree. Then part (σ) is the set of all partitions of finite sets. An element of part (σ) of degree n and part number k is a partition of an n-set into k blocks. Now since $\langle L \mid x^n \rangle = 1$ for all $n > 0$, we conclude that $L = e^A - \epsilon = \epsilon_1 - \epsilon$, the forward difference functional. Thus Proposition 14.1 implies that the number $b_{n,k}$ of partitions of an n-set into k blocks is $S(n, k)$, the Stirling numbers of the second kind, defined by

$$S(n, k) = \frac{(-1)^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n.$$

Corollary 1 tells us that the Bell numbers B_n of partitions of an n-set satisfy

$$B_n = \langle e^L \mid x^n \rangle,$$

where $L = e^A - \epsilon$. Corollary 2 gives the exponential generating function for the Bell numbers as $\exp(e^t - 1)$.

EXAMPLE 2. Let S be an n-set. Then a *k-partition of* S *with selected subsets* is a partition of S into k blocks $B_1, B_2, \ldots, B_k$ together with a nonempty subset C_i of each block B_i.

PROBLEM. Count the number of partitions of an n-set with selected subsets.

Solution. Let σ be the store consisting of all pairs B_i, C_i of nonempty sets, such that $B_i \supseteq C_i$. Then part (σ) is the set of all partitions of finite sets with selected subsets. Since $\langle L \mid x^n \rangle = 2^n - 1$, we see that

$$L = \sum_{k \geqslant 1} (2^k - 1) A^k/k! = e^A(e^A - 1),$$

the difference–Abel functional. Applying Proposition 14.1, we find that the number $b_{n,k}$ of k-partitions of an n-set with selected subsets is the kth coefficient of the nth conjugate Gould polynomial:

$$b_{n,k} = \sum_{i=0}^{n} \binom{n}{i} k^i S(n-i, k).$$

By Corollary 1, the number P_n of partitions with selected subsets is

$$P_n = \langle e^L \mid x^n \rangle,$$

where $L = e^A(e^A - 1)$.

EXAMPLE 3. Find the number of rooted, labeled trees on n vertices, where each vertex is of degree 1, except the root.

Solution. Let σ be the store whose elements of degree n are rooted, labeled trees on n vertices in which each vertex has degree one, except the root. Then part (σ) consists of all forests with the specified degree requirements. Since $\langle L \mid x^n \rangle = n$, we conclude that $L = Ae^A$, the Abel functional. Proposition 14.1 implies that the number of forests on n vertices with k components satisfying the above degree requirements is

$$b_{n,k} = \binom{n}{k} k^{n-k}.$$

Corollary 2 implies that the exponential generating function for the number c_n of forests on n vertices with the above degree requirements is $\exp(te^t)$.

EXAMPLE 4. Find the number of permutations of an n-set all of whose cycles have odd cardinality.

Solution. Let σ be the store whose elements of degree $2n+1$ are all cyclic permutations of the set $\{1, 2, \ldots, 2n+1\}$, and having no elements of even degree. Then part (σ) is the set of permutations of finite sets whose cycles have odd cardinality, the elements of degree $2n+1$ and part number k being permutations of the set $\{1, 2,..., 2n+1\}$ with k cycles, all of odd cardinality. Thus $\langle L \mid x^{2n+1}\rangle = (2n)!$ and $\langle L \mid x^{2n}\rangle = 0$. Hence

$$L = \sum_{k\geqslant 0} \frac{(2k)!}{(2k+1)!} A^{2k+1} = \sum_{k\geqslant 0} \frac{1}{2k+1} A^{2k+1}$$

$$= \tfrac{1}{2}\log((\epsilon + A)/(\epsilon - A)) = \text{arc tanh } A$$

Now let $T_n(x)$ be the conjugate polynomials for L. The recurrence

$$T_{n+1}(x) = xT_n(x) + n(n-1)\, T_{n-1}(x)$$

is established by applying $\tilde{L}^k$ (where $\tilde{L} = \tanh A$) to both sides and using the fact that $\langle \tilde{L}^k \mid T_n(x)\rangle = n!\, \delta_{n,k}$ and

$$\langle \tilde{L}^k \mid xT_n(x)\rangle = \langle \partial_A(\tilde{L}^k) \mid T_n(x)\rangle = \langle k\tilde{L}^{k-1}(\epsilon - \tilde{L}^2) \mid T^n(x)\rangle.$$

Putting $x = 1$ in the above recurrence, we have

$$T_{n+1}(1) = T_n(1) + n(n-1)\, T_{n-1}(1)$$

so that the required number is

$$T_n(1) = 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2.$$

EXAMPLE 5 (Cayley). Find all rooted, labeled trees with n vertices.

Solution. Let σ be the store whose elements of degree n are all rooted labeled trees on n vertices. Then part (σ) is the set of forests. Letting $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, we have the obvious recursion, obtained by removing the root of a tree and counting the resulting forest:

$$b_{n,1} = n \sum_k b_{n-1,k}\,.$$

In terms of the generating functional, this becomes

$$\langle L \mid x^n\rangle = n \sum_k \frac{\langle L^k \mid x^{n-1}\rangle}{k!} = \langle e^L \mid Dx^n\rangle$$

$$= \langle Ae^L \mid x^n\rangle,$$

and thus $L = Ae^L$. We seek the conjugate sequence for L. But $A = Le^{-L} = f(L)$ and so $L = f^{-1}(A)$ and $\tilde{L} = f(A) = Ae^{-A}$. Thus we see that $\tilde{L}$ is the Abel functional and

$$\sum_k b_{n,k}x^k = x(x+n)^{n-1}.$$

Therefore,

$$b_{n,k} = \binom{n-1}{k-1} n^{n-k},$$

and the numbers of rooted labeled trees on n vertices is n^{n-1}.

Recall that a binary tree is a tree in which each vertex has degree one or three, except the root, which has degree 2.

EXAMPLE 6. Find all rooted, labeled binary trees with n vertices.

Solution. Let σ be the store whose elements of degree n are binary trees with n vertices. Then part (σ) is the set of forests of such trees. We have as in Example 3:

$$\langle L \mid x^n \rangle = n\langle L^2 \mid x^{n-1} \rangle + \langle A \mid x^n \rangle,$$

so $L = A(L^2 + 2\epsilon)/2$ and $\tilde{L} = 2A/(A^2 + 2\epsilon)$. By the Transfer Formula,

$$\begin{aligned}\sum_k b_{n,k} x^k &= x2^{-n}(D^2+2)^n x^{n-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^{-k}(n-1)_{2k}\, x^{n-2k}.\end{aligned}$$

Thus,

$$b_{n,k} = \begin{cases} 0, & n+k \text{ odd}. \\ \binom{n}{(n-k)/2} (n-1)_{n-k}\, 2^{(k-n)/2}, & n+k \text{ even}. \end{cases}$$

A linearly ordered tree is one in which all but two vertices are of degree 2.

EXAMPLE 7. Find all rooted, labeled forests on n vertices in which each tree is linearly ordered.

Solution. Let σ be the store in which the elements of degree n are rooted, labeled linearly ordered trees on n vertices. As in Example 3, we see by removing the root that

$$\langle L \mid x^n \rangle = n\langle L \mid x^{n-1} \rangle + \langle A \mid x^n \rangle.$$

Thus $L = LA + A$, and $\tilde{L} = A/(A + \epsilon)$. By the Transfer Formula,

$$\begin{aligned}\sum_k b_{n,k} x^k &= x(D+I)^n x^{n-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} (n-1)_{k-1}\, x^k \\ &= L_n(-x),\end{aligned}$$

where $L_n(x)$ are the Laguerre polynomials.

Bibliography

References cited in the following four works will not be repeated here.

R. Mullin and G.-C. Rota, Theory of binomial enumeration, *in* "Graph Theory and Its Applications," Academic Press, New York/London, 1970.

G.-C. Rota, D. Kahaner, and A. Odlyzko, Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 685–760.

G.-C. Rota, "Finite Operator Calculus," Academic Press, New York/London, 1975.

G.-C. Rota, Hopf algebra methods in combinatorics, to appear.

J. Aczel, Functions of binomial type mapping groupoids into rings, *Math. Z.*, to appear.

S. Agou, "Formules explicités intervenant dans la division Euclidienne des polynômes à coéfficients dans un anneau unitaire et applications diverses," Publications du Département de Mathématiques, Lyon 1971.

W. A. Al-Salam and M. E. H. Ismail, Some operational formulas, *J. Math. Anal. Appl.* **51** (1975), 208–218.

W. A. Al-Salam and L. Carlitz, Generalized Turán expressions for certain hypergeometric series, *Portugal. Math.* **16** (1957), 119–127.

M. G. Arsove, The Pincherele basis problem and a theorem of Boas, *Math. Scand.* **5** (1957), 271–275.

M. C. Arsove, Proper bases and linear homeomorphisms in spaces of analytic functions, *Math. Ann.* **135** (1958), 235–243.

M. G. Arsove, The Paley–Wiener theorem in metric linear spaces, *Pacific J. Math.* **10** (1960), 365–379.

M. G. Arsove, On the behavior of Pincherele basis functions, *Pacific J. Math.* **44** (1973), 13–31.

L. Berg, Zur Umkehrung "periodischer" Operatoren, *Zamm* **55** (1975), 65–68.

L. Berg, Konstruktion von Operatorenkörpern, *Zamm* **44** (1964).

L. Berg, Bemerkungen zurn Interpolationsproblem, *Wiss. Z. Hochsch. Elektrotech. Ilmenau* 2 (1956), 155–157.

L. Berg, Asymptotische Entwicklung einer klasse von Integralen, *Zamm* **36** (1956).

L. Berg, Ein neuer Beweis der Eulerschen Summenformel, *Wiss. Z. Univ. Halle* **9** (1960), 281–284.

L. Berg, Randwertprobleme in der Operatorenrechnung, *Wiss. Z. Univ. Rostock* **23** (1974), 621–624.

L. Berg, Evaluation of the excessive initial values in the two dimensional operational calculus, *Zamm* **55** (1975), 701–708.

L. Berg, Iterationen von beliebiger Ordnung, *Zamm* **40** (1960), 215–229.

L. Berg, "Operatorenrechnung und Asymptotik," Entwicklung der Math. in der DDR, Berlin, 1974.

L. Berg, Unstetige, monotone Iterationsgruppen reeller Funktionen, *Publ. Math. Debrecen* **9** (1962), 47–56.

L. Berg, Uber gewisse Polynome von Tricomi, *Math. Nachr.* **24** (1962).

L. Berg, Uber eine spezielle folge von Polynomen, *Math. Nachr.* **20** (1959), 152–158.

L. Berg, Allegemeine operatorenrechnung, *Uberblicke Math.* **6** (1973), 1–49.

L. Berg, Uber die Struktur der losungen homogener Operatorgleichungen, *Math. Nachr.* **67** (1975), 7–11.

L. Berg, Zur Operatorenrechnung für Funktionen einer diskreten Veränderlichen, *Studia Math.* **20** (1961), 227–243.

L. Berg, Analysis in Geordneten, komnutativen Halbgrupper mit Nullelement, *Nova Acta Leopoldina* **43** (1975), 5–28.

B. C. BERNDT, Elementary evaluation of $\xi(2n)$, *Math. Mag.* **48** (1975), 148–154.

B. C. BERNDT, Character analogues of the Poisson and Euler–MacLaurin summation formulas with applications, *J. Number Theory* **7** (1975), 413–445.

B. C. BERNDT, Periodic Bernoulli numbers, summation formulas and applications, *in* "Theory and Applications of Special Functions," Academic Press, New York/London, 1975.

B. C. BERNDT AND L. SCHOENFELD, Periodic analogues of the Euler–MacLaurin and Poisson formulas with applications to number theory, *Acta Arith.* **28** (1975), 23–68.

G. R. BLAKLEY, Algebra of formal power series, *Duke Math. J.* **31** (1964), 341–345.

G. R. BLAKELY, Combinatorial remarks on partitions of a multipartite number, *Duke Math. J.* **31** (1964), 335–340.

G. R. BLAKELY, Formal solution of nonlinear simultaneous equations: Reversion of series in several variables, *Duke Math. J.* **31** (1964), 347–357.

J. BLISSARD, Theory of generic equations, *Quart. J. Pure Appl. Math.* **4** (1861), 279–305; **5** (1862), 58–75, 184–208.

J. BLISSARD, Note on certain remarkable properties of numbers, *Quart. J. Pure Appl. Math.* **5** (1862), 184.

J. BLISSARD, On the discovery and properties of a peculiar class of algebraic formulae, *Quart. J. Pure Appl. Math.* **5** (1862), 325–335.

J. BLISSARD, Examples of the use and application of representative notation, *Quart. J. Pure Appl. Math.* **6** (1864), 49–64.

J. BLISSARD, On the generalization of certain formulae investigated by Mr. Walton, *Quart. J. Pure Appl. Math.* **6** (1864), 167–179.

J. BLISSARD, Researches in analysis, *Quart. J. Pure Appl. Math.* **6** (1864), 242–257; **7** (1866), 155–170, 223–226.

J. BLISSARD, On the properties of the $\Delta^n 0^n$ class of numbers and of others analogous to them, as investigated by means of representative notation, *Quart. J. Pure Appl. Math.* **8** (1867), 85–110; **4** (1868), 82–94, 154–171.

J. BLISSARD, Note on a certain formula, *Quart. J. Pure Appl. Math.* **4** (1868), 71–76.

J. BLISSARD, On certain properties of the gamma function, *Quart. J. Pure Appl. Math.* **4** (1868), 280–296.

R. P. BOAS, JR., AND H. POLLARD, Continuous analogues of series, *Amer. Math. Monthly* **80** (1973), 18–25.

R. P. BOAS, JR., AND R. C. BUCK, Polynomials defined by generating relations, *Amer. Math. Monthly* **63** (1956), 626–632.

R. P. BOAS, JR., On generalized averaging operators, *Canad. J. Math.* **10** (1957), 122–126.

R. P. BOAS, JR., Exponential transforms and Appell polynomials, *Proc. Nat. Acad. Sci. U.S.A.* **34** (1948), 481–483.

J. W. BROWN, On orthogonal Sheffer sequences, *Glasnik Mat.* **10** (1975), 63–67.

J. W. BROWN, Generalized Appell connection sequences **11**, *J. Math. Anal. Appl.* **50** (1975), 458–464.

W. G. BROWN, On kth roots in power series rings, *Math. Ann.* **170** (1967), 327–333.

J. D. BUCKHOLTZ, Appell polynomial expansions and biorthogonal expansions in Banach spaces, *Trans. Amer. Math. Soc.* **181** (1973), 245–272.

J. D. BUCKHOLTZ, Isomorphism theorems for infinite order differential operators, *Proc. Amer. Math. Soc.* **40** (1973), 533–538.

J. W. BURGMEIER AND R. E. PRATHER, Polynomial calculus with D-like operators, *Amer. Math. Monthly* **82** (1975), 730–737.

TH. BUSK, "On Some General Types of Polynomials," Ejnar Munksgaard, Copenhagen 1955.

L. CARLITZ, The coefficients of sinh x/sin x, *Math. Mag.* (1956), 193–197.

L. Carlitz, The number of coloured graphs, *Canad. J. Math.* **15** (1963), 304–312.

L. Carlitz, Some arithmetic properties of the Olivier functions, *Math. Ann.* **128** (1955), 412–419.

L. Carlitz, Polynomial representations and compositions, I, *Houston J. Math.* **2** (1966), 23–48.

L. Carlitz, A theorem on differential operators, *Amer. Math. Monthly* **83** (1976), 351–354.

L. Carlitz, A special class of triangular arrays, *Collect. Math.* **27** (1976), 3–38.

L. Carlitz, A note on integral-valued polynomials, *Koninkl. Nederl. Akad. Wetensch.* **21** (1959), 294–299.

L. Carlitz and R. Scoville, Tangent numbers and operators, *Duke Math. J.* **39** (1972), 413–429.

L. Carlitz, A binomial identity related to ballots and trees, *J. Combinatorial Theory* **14** (1973), 261–263.

L. Carlitz, Eulerian numbers and polynomials, *Math. Mag.* (1959), 247–260.

L. Carlitz, Note on the integral of the product of several Bernoulli polynomials, *J. London Math. Soc.* **34** (1959), 361–363.

L. Carlitz and J. Riordan, The divided central differences of zero, *Canad. J. Math.* **15** (1963), 94–100.

L. Carlitz, A formula of Bateman, *Proc. Glasgow Math. Assoc.* **3** (1957), 99–101.

L. Carlitz, Transformation of certain bilinear generating functions, *Ann. Mat.* **86** (1970), 155–168.

L. Carlitz, The product of several Hermite or Laguerre polynomials, *Monatsh. Math.* **66** (1962), 393–396.

L. Carlitz, On some polynomials of Tricomi, *Boll. Un. Mat. Ital.* **13** (1958), 58–64.

L. Carlitz, Some formulas of Hermite, *Publ. Faculte Electro. Univ. Belgrade* **303** (1970), 9–13.

L. Carlitz, Some arithmetic properties of a special sequence of polynomials in the Gaussian field, *Duke Math. J.* **30** (1963), 15–24.

P. C. Chakravarti, "Integrals and Sums," Athlone, London, 1970.

J. Cigler, Some remarks on Rota's umbral calculus, to appear.

L. Comtet, Une formule explicité pour les puissances successives de l'opérateur de dérivation de Lie, *C. R. Acad. Sci. Paris* **276** (1973), 165–168.

L. Comtet, Nombres de Stirling généraux et fonctions symétriques, *C. R. Acad. Sci. Paris* **275** (1972), 747–750.

L. Comtet, Sur les dérivés successives d'une fonction implicite, *C. R. Acad. Sci. Paris* **278** (1974), 249–251.

R. Cohen, "Difference Algebra," Interscience, New York, 1965.

J. Delsarte, "Oeuvres de Jean Delsarte," Vols. 1, 2, Centre National de la Recherche Scientifique, Paris, 1971.

V. Dobrowolny, Konstruktion eines Operatorenkorpers zur Mellin-Rücktransformation, *Wiss. Z. Univ. Rostock* **16** (1967), 771–778.

G. Doetsch, Die in der Statistik seltener Ereignisse auftretenden Charlierschen Polynome und eine damit zusammenhangende differential-Differenzenglerchung, *Math. Ann.* **109** (1933), 258–266.

G. Doetsch, Eulerian series and applications, *in* "Abstracts and Problems from the Conference at Pennsylvania State University 1974."

S. Dubuc, Problèmes relatifs à l'itération de fonctions suggères par les processus en cascade, *Ann. Inst. Fourier Univ. (Grenoble)* **21** (1971), 171–251.

A. E. Fekete, "Substitution of Power Series," Department of Mathematics, Memoir, University of Newfoundland, 1965.

W. FELLER, "An Introduction to Probability Theory and Its Applications," Wiley, New York, 1971.

D. FOATA, "La série génératrice exponentielle dans les problèmes d'énumeration," Le Presses de l'Université de Montréal, Montréal, 1974.

R. F. FOX, A generalized theory of multiplicative stochastic processes using cumulant techniques, *J. Math. and Phys.* **16** (1975), 289–297.

J. FRANÇON, Preuves combinatoires des identités d'Abel, to appear.

A. M. GARSIA AND S. A. JONI, Higher dimensional polynomials of binomial type and formal power series, *Comm. Algebra*, to appear.

A. GARSIA, An exposé of the Mullin–Rota theory of polynomials of binomial type, *J. Linear and Multi-linear Algebra* **1** (1973), 47–65.

I. M. GESSEL, "Generating Functions and Enumeration of Sequences," Ph.D. Thesis, M.I.T., 1977.

I. J. GOOD, The generalization of Lagrange's expansion and the enumeration of trees, *Proc. Cambridge Philos. Soc.* **61** (1956), 499–517.

H. W. GOULD, An identity involving Stirling numbers, *Ann. Inst. Statist. Math.* **17** (1965), 265–269.

H. W. GOULD, The Lagrange interpolation formula and Stirling numbers, *Proc. Amer. Math. Soc.* **11** (1960), 421–425.

H. W. GOULD, "Research Bibliography of Two Special Number Sequences," privately printed.

S. GRABINER, A formal power series operational calculus for quasinilpotent operators, *Duke Math. J.* **38** (1971), 641–658.

S. GRABINER, Derivations and automorphisms of Banach algebras of power series, *Mem. Amer. Math. Soc.* **146** (1974).

B. HARRIS AND L. SCHOENFELD, A composition theorem for the enumeration of certain subsets of the symmetric semigroup, *in* "Graph Theory and Its Applications," Academic Press, New York/London, 1970.

H. D. HAYES, The roots of the equation $x = (c \exp)^n x$ and the cycles of the substitution $(x \mid ce^x)$, *Quart. J. Math. Oxford* **2** (1952), 81–90.

M. HENLE, Dissection of generating functions, *Stud. Appl. Math.* **51** (1972), 397–410.

M. HENLE, Binomial enumeration on dissects, *Trans. Amer. Math. Soc.* **202** (1975), 1–39.

P. HENRICI, An algebraic proof of the Lagrange–Bürmann formula, *J. Math. Anal. Appl.* 8 (1964), 218–224.

O. HOLDER, Elementare Herleitung einer dem binomischen satz verwandten Formel, *Ans Ber. Math.* **9** (1936), 133–134.

O. HOLDER, Uber eine Verallgemunerung der binomischen Formel, *Ans Ber. Math.* 18 (1936), 61–66.

F. HOLLAND, An algebraic characterization of the Laguerre functions, *Quart. J. Math.* **20** (1969), 193–194.

A. HURWITZ, "Mathematische Werke," Vols. I, II, Birkhauser Verlag, Basel/Stuttgart, 1962.

M. E. H. ISMAIL AND M. T. RASHED, Polynomial expansions and generating functions, to appear.

W. JENTSCH, Uber die Struktur der rationalen Operatoren in der zweidimensionalen diskereten Operatorenrechnung, *Studia Math.* **17** (1972), 16–27.

S. A. JONI, Lagrange inversion in higher dimensions and umbral operators, *J. Linear and Multi-linear Algebra*, to appear.

L. KOSCHMIEDER, El teorema de adición de los polinomios de Hermite de varias variables, *Rev. Mat. Hisp.-Amer.* **9** (1930), 274–280.

I. KOSCHMIEDER, Observación a un teorema de Fejer, *Rev. Mat. Hisp.-Amer.* **6** (1929).

K.-H. KRAUSE, Uber die Koeffizienten einer speziellen asymptotischen Entwicklung, *Wiss. Z. Univ. Halle* **13** (1964), 439–441.

W. KRULL, Zur Arithmetik der Potenzreihen mit ganzzahligen Koeffizienten, *J. Reine Angew. Math.* **164** (1931), 12–22.

F. LETTENMEYER, Uber das asymptotische Verhalten der Lösungen von Differenzengleichungen und Systemen von Solchen, *Sitz. Bay. Akad. Wiss.* **10** (1930), 73–100.

F. LETTENMEYER, Uber den Koeffizientenkorper von reihenentwichlungen insbesondere algebraischer Funktionen, *Sitz. Bay. Akad. Wiss.* **5** (1933), 103–127.

F. LETTENMEYER, Uber Gleichungen unendlich hochen Grades in zwei Variabeln, *Sitz. Bay. Akad. Wiss.* **8** (1933), 317–371.

B. MAZBIC-KULMA, Differential equations in a linear ring, *Studia Math.* **33** (1971), 155–159.

J. C. MCCONNELL, A note on the Weyl algebra, to appear.

J. C. MCCONNELL AND J. C. ROBSON, Homomorphisms and extensions of modules over certain differential polynomial rings, *J. Algebra* **26** (1973), 319–342.

P. MILCH, A probabilistic proof of a formula for Jacobi polynomials by L. Carlitz, *Proc. Cambridge Philos. Soc.* **64** (1968), 695–698.

P. MILCH, The Jacobi polynomial and some hypergeometric-type distributions, *Ann. Inst. Statist. Math.* **22** (1970), 465–474.

J. MILKMAN, Hermite polynomials, Hermite functionals and their integrals in real Hilbert space, *Riv. Mat. Univ. Parma* **6** (1955), 1–24.

L. M. MILNE-THOMSON, "The Calculus of Finite Differences," Macmillan & Co., London, 1951.

N. NIELSEN, "Gammafunktion," Chelsea, New York, 1965.

H.-J. PRESIA, Asymptotische Entwicklungen für eine Klasse von Laplace-Integralen, *Mitt. Inst. Math.* **8** (1962), 177–187.

A. RAN, One-parameter groups of formal power series, *Duke Math. J.* **38** (1971), 441–459.

R. REDHEFFER, Induced transformations of the derivative-vector, *Amer. Math. Monthly* **83** (1976), 255–259.

D. L. REINER, "The Combinatorics of Polynomial Sequences," Ph.D. Thesis, University of Minnesota, 1974.

R. RIVET, Sur le corps des fractions d'un auneau de series formelles, *C. R. Acad. Sci. Paris* **273** (1971), 1–3.

R. RIVET, Anneaux de series formelles et anneaux henseliens, *C.I.M.E.* **12** (1971), 199–210.

D. W. ROBINSON, Separable endomorphisms and higher-order commutators, *Linear Algebra and Appl.* **4** (1971), 285–296.

N. SANKARAN, Rings of formal power series, *Canad. Bull. Math.* **14** (1971), 207–220.

S. SCHEINBERG, Power series in one variable, *J. Math. Anal. Appl.* **31** (1970), 321–333.

R. V. SKARDA, JR., "Groups of Power Series under Composition," preprint.

H. M. SRIVASTAVA AND R. G. BUSHMAN, Some polynomials defined by generating relations, *Trans. Amer. Math. Soc.* **205** (1975), 360–370.

R. P. STANLEY AND I. GESSEL, Stirling polynomials, to appear.

S. M. TANNY, On some numbers related to the Bell numbers, *Canad. Math. Bull.* **17** (1975), 733–738.

M. TASCHE, Algebraische Operatorenrechrung fur einen rechtsunvertierbaren Operator, *Wiss. Z. Univ. Rostock* **9** (1974), 735–744.

L. TOSCANO, Generalizzazioni dei polinomi Laguerre e dei polinomi attuariali, *Riv. Mat. Univ. Parma* **11** (1970), 191–226.

L. TOSCANO, Sur la somme de deux séries du type intégrogéométriques et exponentielle généralisées, *An. Fac. Ci. Univ. Porto* **54** (1971), 1–12.

L. Toscano, Contribution à l'étude de polynômes associés aux fonctions du cylindre parabolique, *An. Fac. Ci. Univ. Porto* **54** (1971), 1–14.

L. Toscano, Déterminants et polynômes de Laguerre généralisés, *Publ. Inst. Math.* **10** (1970), 65–70.

L. Toscano, Su gli operatori permatabili di secondo ordine, *Giorn. Mat. Battaglini* **90** (1962), 1–17.

L. Toscano, Sulle funzioni del cilindro parabolico, *La Mat.* **26** (1971), 1–23.

L. Toscano, Relazioni tra polinomi classici, atti dell'Instituto Veneto di Scienze, *Lettere ed arti* **131** (1972), 267–287.

L. Toscano, Sur deux suites recurrentes du seconde ordre linéaires et homogènes, *Mat. Bech.* **9** (1972), 131–140.

L. Toscano, Operatori permutabili di secondo ordine, *Rend. Accad. Nay. Lincei* **23** (1936), 309–312.

L. Toscano, Operatori q-permutabili di secondo ordine, *Ann. Scolastico* (1962), 6–38.

L. Toscano, Una classe di polinomi della matematica attuariale, *Riv. Mat. Univ. Parma* **1** (1950), 459–470.

M. Ward, The reversion of a power series, *Rend. Circ. Mat.* **54** (1930), 1–5.

E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," Cambridge Univ. Press, London/New York, 1952.

N. Zierler and W. H. Mills, Products of linear recurring sequences, *J. Algebra* **27** (1973), 147–157.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.