unless $n = jk$, and

$$B_{jk,k} = \frac{(jk)!}{k!\,(j!)^k}\, x_j\,.$$

(e) Every delta functional can be obtained from the Bell generic delta functional by specializing the values of the x_i. Thus every formula for the Bell polynomials gives a formula for all conjugate sequence. For example, from (b) one obtains

$$\langle L^k \mid x^n \rangle = \sum_{j=0}^{k} \binom{k}{j} (n)_k \langle L \mid x \rangle^{k-j} \langle L_1{}^j \mid x^{n-k} \rangle,$$

where L is any delta functional and where $L = L_1 A + x_1 A$. Similarly, (c) gives the conjugate polynomials of the sum of two (or more) delta functionals in terms of the conjugate sequences of the summand.

6. Automorphisms and Derivations

Given two polynomial sequences $p_n(x)$ and $q_n(x)$, a frequently encountered problem is that of determining a matrix of constants $c_{n,k}$, which we call the *connection constants* of $p_n(x)$ with $q_n(x)$, such that

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} p_k(x). \qquad (*)$$

In this section, we give a solution to this problem when the polynomial sequences are of binomial type. The solution we propose takes a particularly simple form in the *umbral notation* we now introduce. If $r(x) = \sum_{k=0}^{n} c_k x^k$ is a polynomial, and $p_n(x)$ is a polynomial sequence, the *umbral composition* of $r(x)$ with $p_n(x)$ is the polynomial, written $r(\mathbf{p}(x))$, and defined by

$$r(\mathbf{p}(x)) = \sum_{k=0}^{n} c_k p_k(x).$$

If $r_n(x)$ and $p_n(x)$ are two polynomial sequences, the umbral composition of $r_n(x)$ with $p_n(x)$ is the polynomial sequence $r_n(\mathbf{p}(x))$. In this notation, $(*)$ becomes

$$q_n(x) = r_n(\mathbf{p}(x)),$$

where $r_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$.

Umbral composition is simply the result of applying a suitable linear operator to a polynomial sequence. In particular, if α is the linear operator on P defined by $\alpha x^n = p_n(x)$ for $n = 0, 1, 2, \ldots$, then $\alpha r_n(x) = r_n(\mathbf{p}(x))$, and $(*)$ becomes

$$q_n(x) = \alpha r_n(x).$$

Thus the constants $c_{n,k}$ are determined once the polynomials $r_n(x) = \alpha^{-1}q_n(x)$ are known.

We are therefore led to define an *umbral operator* as a linear operator α on P, given by $\alpha x^n = p_n(x)$, where $p_n(x)$ is a sequence of binomial type. When we wish to emphasize the delta functional L for which $p_n(x)$ is the associated sequence, we write α_L for α.

Before proceeding further, we recall some basic facts about adjoints of linear operators. Let T be a linear operator mapping P into itself. The adjoint T^* of T is the operator mapping P^* into itself uniquely defined by

$$\langle T^*(L) \mid p(x)\rangle = \langle L \mid Tp(x)\rangle$$

for all $L \in P^*$ and all $p(x) \in P$. The adjoint T^* of a linear operator T on P exists and is continuous. To see the latter, suppose L_n is a sequence of linear functionals converging to L. For any polynomial $p(x)$, we have

$$\langle T^*(L_n) \mid p(x)\rangle = \langle L_n \mid Tp(x)\rangle,$$

and by the definition of convergence in P^*, if n is large, this equals

$$\langle L \mid Tp(x)\rangle = \langle T^*(L) \mid p(x)\rangle.$$

Thus $T^*(L_n)$ converges to $T^*(L)$, and T^* is continuous.

On the other hand, suppose U is a linear operator mapping P^* into itself. Then the adjoint U^* maps P^{**} into itself. Thinking of P as a subspace of P^{**}, in general U^* will not map P into itself. The sufficient condition to ensure that U^* maps polynomials to polynomials is the continuity of U. We have

PROPOSITION 6.1. *A linear operator mapping P^* into itself is the adjoint of a linear operator mapping P into itself if and only if it is continuous.*

Proof. We have already seen that the adjoint of an operator mapping P into itself is continuous. For the converse, suppose U is a continuous operator mapping P^* into itself. Since the sequence of powers A^k converges to zero, so does the sequence $U(A^k)$. Thus the function

$$p_n(x) = \sum_{k=0}^{\infty} \frac{\langle U(A^k) \mid x^n\rangle}{k!} x^k$$

is a polynomial, and

$$\langle A^k \mid p_n(x)\rangle = \langle U(A^k) \mid x^n\rangle$$

for all $k \geqslant 0$.

If we define the operator V mapping P into itself by $Vx^n = p_n(x)$, then

$$\begin{aligned}\langle V^*(L) \mid x^n\rangle &= \langle L \mid p_n(x)\rangle \\ &= \langle U(L) \mid x^n\rangle, \qquad \text{for all } L \in P^*,\end{aligned}$$

the last equality by the spanning argument for linear functionals. Thus $V^*(L) = U(L)$ for all $L \in P^*$ and so $V^* = U$.

We return now to the main stream of this section. The *shift* of a polynomial sequence $p_n(x)$ is the operator θ, mapping P into itself, defined by $\theta p_n(x) = p_{n+1}(x)$. If $p_n(x)$ is of binomial type, we say that θ is an *umbral shift*. By θ_L, we mean the umbral shift defined by the associated sequence for L.

Umbral operators and umbral shifts are related to automorphisms and derivations of the umbral algebra. Recall that a derivation ∂ of the umbral algebra is a linear operator such that $\partial(LM) = (\partial L)M + L(\partial M)$.

In order to exhibit the aforementioned relationship, we require two lemmas.

LEMMA 1. *Any continuous automorphism of the umbral algebra maps delta functionals to delta functionals.*

Proof. Let β be a continuous automorphism of P^*, and let L be a delta functional. By Proposition 3.6, $\langle L \mid 1\rangle = 0$ implies L^n converges to zero. The continuity of β implies that $\beta(L^n) = \beta(L)^n$ converges to zero, and another application of Proposition 3.6 implies $\langle \beta(L) \mid 1\rangle = 0$. By Proposition 4.2, the powers of L span P^*, and thus so do the powers of $\beta(L)$. The same proposition implies $\beta(L)$ is a delta functional.

LEMMA 2. *Let ∂ be a derivation of the umbral algebra which is everywhere defined, continuous, and onto. Then there is a delta functional L such that $\partial L = \epsilon$.*

Proof. Since ∂ is onto, there is a linear functional L for which $\partial L = \epsilon$. Since ∂ is a derivation, we infer that $\partial\epsilon = 0$, hence, subtracting from L a constant if necessary, we may assume that $\langle L \mid 1\rangle = 0$. Now we expand L into a series of powers of the generator $a_1A + a_2A^2 + \cdots$, and since ∂ is continuous, we may apply it term by term to the series. Since $\partial A^n = nA^{n-1}\,\partial A$, we have

$$\epsilon = (a_1 + 2a_2A + \cdots)\,\partial A.$$

Thus the series $(a_1 + 2a_2A + \cdots)$ is invertible and so $a_1 \neq 0$. That is, $\langle L \mid x\rangle \neq 0$ and L is a delta functional.

We are now ready to prove

THEOREM 5. (a) *An operator α of P onto itself is an umbral operator if and only if its adjoint α^* is a continuous automorphism of the umbral algebra.*

(b) *An operator θ of P into itself is an umbral shift if and only if its adjoint θ^* is a continuous, everywhere defined derivation of the umbral algebra onto itself.*

Proof. (a) It is clear that the adjoint α^* of an umbral operator α is linear, continuous, one-to-one, and onto. Thus all that remains is to show that α^* preserves multiplication. Letting $\alpha x^n = p_n(x)$, this follows from the spanning argument and the following calculations:

$$\begin{aligned}\langle \alpha^*(MN) \mid x^n\rangle &= \langle MN \mid \alpha x^n\rangle = \langle MN \mid p_n(x)\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle M \mid p_k(x)\rangle\langle N \mid p_{n-k}(x)\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle \alpha^*(M) \mid x^k\rangle\langle \alpha^*(N) \mid x^{n-k}\rangle \\ &= \langle \alpha^*(M)\,\alpha^*(N) \mid x^n\rangle.\end{aligned}$$

For the converse, suppose β is a continuous automorphism of the umbral algebra. In view of Lemma 1, we may let $p_n(x)$ be the associated sequence for the delta functional $L = \beta^{-1}(A)$. Defining the umbral operator α by $\alpha x^n = p_n(x)$, we have

$$\langle \beta(L)^k \mid x^n\rangle = n!\,\delta_{n,k} = \langle L^k \mid \alpha x^n\rangle,$$

for all k and n. By the Expansion Theorem, the same identity holds for all linear functionals M,

$$\langle \beta(M) \mid x^n\rangle = \langle M \mid \alpha x^n\rangle,$$

and thus by the spanning argument, $\beta(M) = \alpha^*(M)$. Hence part (a) is proved.

(b) Let θ be the umbral shift defined by $\theta p_n(x) = p_{n+1}(x)$, and suppose $p_n(x)$ is the associated sequence for L. We have seen that the adjoint of a linear operator on P is continuous. Moreover,

$$\langle L^k \mid p_{n+1}(x)\rangle = k\langle L^{k-1} \mid p_n(x)\rangle$$

and thus

$$\langle \theta^*(L^k) \mid p_n(x)\rangle = k\langle L^{k-1} \mid p_n(x)\rangle,$$

for all $n, k \geqslant 0$. Therefore, by the spanning argument $\theta^*(L^k) = kL^{k-1}$ and so θ^* is an everywhere defined derivation, and is onto.

Conversely, let ∂ be a continuous derivation of the umbral algebra onto itself. By virtue of Lemma 2, we may let $p_n(x)$ be the associated sequence for the delta functional L, with $\partial(L) = \epsilon$. Then for $k \geqslant 0$, we have

$$\begin{aligned}\langle L^k \mid \partial^* p_n(x)\rangle &= \langle \partial(L^k) \mid p_n(x)\rangle \\ &= k\langle L^{k-1} \mid p_n(x)\rangle = k(k-1)!\,\delta_{k-1,n} \\ &= k!\,\delta_{k,n+1}\,.\end{aligned}$$

By the uniqueness of the associated sequence, it follows that $\partial^* p_n(x) = p_{n+1}(x)$, and part (b) is proved.

Every continuous automorphism β of the umbral algebra is thus associated with a unique delta functional L, namely, the delta functional whose associated polynomials are $p_n(x) = \beta^* x^n$. Similarly, every continuous, everywhere defined derivation β of the umbral algebra onto itself is associated with a unique delta functional L, the one for which ∂^* is the umbral shift of the associated sequence. We shall stress this association by writing $\beta = \beta_L$ and $\partial = \partial_L$. We remark that $\beta_L(L) = A$ and $\partial_L(L) = \epsilon$.

As an example, the simplest umbral operator is the substitution $x^n \to a^n x^n$, for $a \in K$. Its adjoint maps A^k to $a^k A^k$. The simplest shift is the map θ_A: $x^n \to x^{n+1}$, and its adjoint ∂_A is

$$\langle \partial_A L \mid p(x) \rangle = \langle L \mid xp(x) \rangle.$$

We proceed to develop some corollaries of Theorem 5.

COROLLARY 1. (a) *An umbral operator maps sequences of binomial type to sequences of binomial type.*

(b) *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, and if α is an operator defined by $\alpha p_n(x) = q_n(x)$, then α is an umbral operator.*

(c) *If $p_n(x)$ is the associated sequence for L and $q_n(x)$ is the associated sequence for M, then the adjoint of the umbral operator $\alpha p_n(x) = q_n(x)$ satisfies $\alpha^*(M) = L$.*

(d) *If ∂_L is the derivation associated with the delta functional L, then $\partial_L M = 0$ if and only if $M = a\epsilon$, for some $a \in K$.*

Proof. (a) Suppose α is an umbral operator, and $q_n(x)$ is a sequence of binomial type, with associated delta functional M. Then

$$\begin{aligned}\langle (\alpha^{-1})^*(M)^k \mid \alpha q_n(x) \rangle &= \langle M^k \mid \alpha^{-1} \alpha q_n(x) \rangle \\ &= \langle M^k \mid q_n(x) \rangle = k!\, \delta_{n,k}.\end{aligned}$$

Thus $\alpha q_n(x)$ is the associated sequence for the delta functional $(\alpha^{-1})^* M$, and is therefore of binomial type.

(b) A slight modification of the calculations in the proof of Theorem 5 will show that, if $\alpha p_n(x) = q_n(x)$, then α^* is a continuous automorphism of the umbral algebra, and thus α is an umbral operator.

(c) This follows by noticing that

$$\begin{aligned}\langle \alpha^*(M) \mid p_n(x) \rangle &= \langle M \mid \alpha p_n(x) \rangle \\ &= \langle M \mid q_n(x) \rangle = \delta_{n,1} = \langle L \mid p_n(x) \rangle\end{aligned}$$

for all $n \geqslant 0$.

(d) Clearly, $\partial_L(a\epsilon) = 0$. The converse follows by observing that, for $p_n(x)$ the associated sequence for L, $0 = \langle \partial_L M \mid p_n(x)\rangle = \langle M \mid p_{n+1}(x)\rangle$, and thus $M = a\epsilon$ for some $a \in K$.

Part (a) of the preceding corollary implies that the composition of two umbral operators is an umbral operator. This allows us to define a group operation on delta functionals, which we call *composition*, as follows. If L and M are two delta functionals with associated umbral operators α_L and α_M, the composition $L \circ M$ is the delta functional associated with the umbral operator $\alpha_L \circ \alpha_M$.

PROPOSITION 6.2. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for L and M, respectively, then $q_n(\mathbf{p}(x))$ is of binomial type, being the associated sequence for $L \circ M$.*

Proof. Since $\alpha_L: x^n \to p_n(x)$ and $\alpha_M: x^n \to q_n(x)$, it follows that $\alpha_L \circ \alpha_M: x^n \to q_n(\mathbf{p}(x))$. Since $\alpha_L \circ \alpha_M$ is an umbral operator, $q_n(\mathbf{p}(x))$ is of binomial type, and is the associated sequence of $L \circ M$ by definition of composition of delta functionals.

Since the umbral operator α_A is the identity, the generator A is the identity under composition of delta functionals, and thus $L \circ A = A \circ L = L$ for all delta functionals L.

Recall that we defined the delta functional M to be reciprocal to the delta functional L whenever the associated sequence for L is the conjugate sequence for M.

PROPOSITION 6.3. *A delta functional M is reciprocal to a delta functional L if and only if $L \circ M = A$.*

Proof. Suppose M is reciprocal to L, and let L have associated sequence $p_n(x)$. Then since $p_n(x)$ is the conjugate sequence for M, we have

$$p_n(x) = \sum_{k=0}^{n} \frac{\langle M^k \mid x^n\rangle}{k!} x^k$$

and, by the spanning argument, for any polynomial $q(x)$,

$$q(\mathbf{p}(x)) = \sum_{k=0}^{\infty} \frac{\langle M^k \mid q(x)\rangle}{k!} x^k.$$

If we take $q(x) = q_n(x)$, an associated polynomial for M, we find

$$q_n(\mathbf{p}(x)) = x^n.$$

Therefore, $L \circ M = A$ by Proposition 6.2. The converse is obvious.

We remark that if $\alpha_L^{-1} = \alpha_M$ then $\alpha_M \circ \alpha_L = I$ and $M \circ L = A$. Thus by the previous proposition $M = \tilde{L}$, and therefore $\alpha_L^{-1} = \alpha_{\tilde{L}}$.

We are now able to give the connection constants for sequences of binomial type.

PROPOSITION 6.4. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for L and M, respectively, and if*

$$q_n(x) = r_n(\mathbf{p}(x))$$

for a polynomial sequence $r_n(x)$, then $r_n(x)$ is of binomial type, and is the associated sequence for the delta functional $\tilde{L} \circ M$.

Proof. The proof is immediate from Proposition 6.2.

We now interpret the composition of delta functionals in terms of their indicators. Recall that, if $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ is any formal power series in $\mathbf{F}$, and $g(t)$ is any formal power series with zero constant term, then the series

$$f(g(t)) = a_0 + a_1 g(t) + a_2 (g(t))^2 + \cdots,$$

called the composition of $f(t)$ with $g(t)$, converges in the topology of $\mathbf{F}$. In particular, if $g(t)$ is any formal power series whose constant term is zero and whose linear term is nonzero, then there exists a unique formal power series $g^{-1}(t)$, called the inverse of $g(t)$, with the property that $g(g^{-1}(t)) = g^{-1}(g(t)) = t$.

Finally, recall that every formal power series $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ has a derivative $f'(t)$, obtained by termwise differentiation; that is, $f'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots$.

THEOREM 6. *Let L and M be delta functionals, with indicators $f(t)$ and $g(t)$, respectively. Then the composition $L \circ M$ is a delta functional with indicator $g(f(t))$.*

Proof. Writing β_L, β_M, and $\beta_{L \circ M} = \beta_M \circ \beta_L$ for the automorphisms of P^* associated with the delta functionals L, M and $L \circ M$, respectively, we have $\beta_L f(A) = A$, $\beta_M g(A) = A$ and $\beta_{L\circ M}(L \circ M) = A$. Thus $L \circ M = \beta_{L\circ M}^{-1}(A) = (\beta_M \circ \beta_L)^{-1}(A) = \beta_L^{-1} \circ \beta_M^{-1}(A) = \beta_L^{-1} g(A) = g(\beta_L^{-1} A) = g(f(A))$. Therefore the indicator of $L \circ M$ is $g(f(t))$.

COROLLARY 1. *Two delta functionals L and $\tilde{L}$ are reciprocals if and only if their indicators are inverse formal power series.*

We can now include indicators in our solution of the connection constants problem.

PROPOSITION 6.5. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for $L = f(A)$ and $M = g(A)$, respectively, and if*

$$q_n(x) = r_n(\mathbf{p}(x))$$

for a polynomial sequence $r_n(x)$, then $r_n(x)$ is the associated sequence for the delta functional $\tilde{L} \circ M = g(f^{-1}(A))$.

We conclude this section with two results on derivations. The chain rule for derivations of the umbral algebra is easily derived:

PROPOSITION 6.6. *Let ∂_L and ∂_M be the derivations of the umbral algebra associated with the delta functionals L and M, respectively. Then*

$$\partial_L = (\partial_L M)\, \partial_M .$$

Proof. Any linear functional N can be expanded into a convergent series of powers of M:

$$N = a_0 + a_1 M + a_2 M^2 + \cdots, \qquad a_i \in K,$$

and since ∂_L and ∂_M are continuous, we have

$$\partial_L N = a_1 + 2a_2 M \partial_L M + 3a_3 M^2 \partial_L M + \cdots = (\partial_L M)\, \partial_M N.$$

Thus

$$\partial_L = (\partial_L M)\, \partial_M .$$

The following proposition is immediate.

PROPOSITION 6.7. *If L is any delta functional and M is any linear functional, the L-indicator of $\partial_L M$ is the derivative of the L-indicator of M.*

7. SHIFT-INVARIANT OPERATORS

On the algebra of all linear operators on P we define a topology by specifying that a sequence T_k of operators converges to an operator T whenever, given a polynomial $p(x)$, there is an index n_0 such that if $n \geqslant n_0$ then $T_n\, p(x) = Tp(x)$. Under this topology, the algebra of all linear operators is a complete topological algebra.

Every linear functional L defines a *multiplication operator* on P^*, mapping the linear functional M to the linear functional $L \cdot M$. We denote this operator by $\mu(L)^*$. Thus, $\mu(L)^* M = L \cdot M$. Every multiplication operator is continuous, hence by Proposition 6.1, its adjoint $\mu(L)$ maps polynomials into polynomials. In symbols,

$$\langle LM \mid p(x)\rangle = \langle \mu(L)^* M \mid p(x)\rangle = \langle M \mid \mu(L)\, p(x)\rangle.$$

We investigate the properties of the map $L \to \mu(L)$, beginning with

PROPOSITION 7.1. *The mapping $L \to \mu(L)$ of linear functionals into linear operators is a continuous algebra monomorphism.*

Proof. Only the continuity need be verified. Let the sequence L_k of linear functionals converge to zero. Given $n \geqslant 0$, we have $\langle L_k \mid x^j \rangle = 0$ for $j = 0, 1,..., n$ and for large k, depending on n. Hence, for all scalars $a \in K$ and for large k, $\langle L_k \mid (x + a)^n \rangle = 0$ and thus $0 = \langle L_k \mid (x + a)^n \rangle = \langle \epsilon_a \mid \mu(L_k) x^n \rangle$. Therefore $\mu(L_k) x^n = 0$ for k large. Q.E.D.

The set of all operators of the form $\mu(L)$, for some linear functional L, is thus a topological algebra. We call an operator of this form a *shift-invariant operator*, and denote the algebra of all shift-invariant operators by Σ. Thus, the umbral algebra and the algebra of shift-invariant operators on P are isomorphic as topological algebras.

COROLLARY 1. *A shift-invariant operator T is invertible if and only if $T1 \neq 0$.*

As an example, consider the shift-invariant operator $E^a = \mu(\epsilon_a)$. From

$$\langle \epsilon_{a+b} \mid p(x) \rangle = \langle \epsilon_a \epsilon_b \mid p(x) \rangle = \langle \epsilon_b \mid E^a p(x) \rangle$$

we conclude that $E^a p(x) = p(x + a)$. We call E^a the *translation operator*. In particular, $E^0 = I$, the identity of Σ. Similarly, it is seen that $D = \mu(A)$ is the ordinary derivative $Dp(x) = p'(x)$.

A characterization of shift-invariant operators is

PROPOSITION 7.2. *A linear operator T is shift-invariant if and only if $TE^a = E^a T$ for all $a \in K$, that is, if and only if it commutes with all translation operators.*

Proof. Suppose $TE^a = E^a T$ for all $a \in K$ and for some operator T on P. We show that $T = \mu(L)$, where L is the linear functional defined by $\langle L \mid p(x) \rangle = \langle \epsilon \mid Tp(x) \rangle$. In fact:

$$\begin{aligned} \langle \epsilon_a \mid \mu(L)\, p(x) \rangle &= \langle L \epsilon_a \mid p(x) \rangle \\ &= \langle L \mid p(x + a) \rangle = \langle \epsilon \mid Tp(x + a) \rangle \\ &= \langle \epsilon \mid E^a T p(x) \rangle = \langle \epsilon_a \mid Tp(x) \rangle \end{aligned}$$

for all $a \in K$ and thus $\mu(L)\, p(x) = Tp(x)$.

Another characterization of shift-invariant operators is

PROPOSITION 7.3. *Let M be a delta functional. Then a linear operator T is shift-invariant if and only if $T\mu(M) = \mu(M)T$.*

The proof is omitted.

In view of the isomorphism between the algebras P^* and Σ, we may expect operator analogs of some of the notions introduced for the study of the umbral algebra.

A *delta operator* is an operator of the form $Q = \mu(L)$, where L is a delta functional. Delta operators are characterized by the following property of immediate verification:

PROPOSITION 7.4. *A shift-invariant operator Q is a delta operator if and only if $Q1 = 0$ and Qx is a nonzero constant.*

If $Q = \mu(L)$ is a delta operator, the *associated sequence for Q* is defined to be associated sequence for L. The relationship between a delta operator and its associated sequence $p_n(x)$ is a generalization of the relationship between the derivative operator and the sequence $p_n(x) = x^n$.

PROPOSITION 7.5. *The polynomial sequence $p_n(x)$ is the associated sequence for the delta operator Q if and only if it satisfies the following conditions:*

(i) $p_0(x) = 1$,

(ii) $p_n(0) = 0$ *for* $n > 0$,

(iii) $Qp_n(x) = np_{n-1}(x)$.

Proof. Let $Q = \mu(L)$ and suppose first that $p_n(x)$ is the associated sequence for Q, and hence for L. Then

$$\begin{aligned}\langle L^k \mid Qp_n(x)\rangle &= \langle L^{k+1} \mid p_n(x)\rangle \\ &= n!\, \delta_{k+1,n} = \langle L^k \mid np_{n-1}(x)\rangle.\end{aligned}$$

Therefore, by the Expansion Theorem,

$$\langle M \mid Qp_n(x)\rangle = \langle M \mid np_{n-1}(x)\rangle$$

for every linear functional M, and thus $Qp_n(x) = np_{n-1}(x)$.

Conversely, suppose the polynomial sequence $p_n(x)$ satisfies (i), (ii), and (iii). Then

$$\begin{aligned}\langle L^k \mid p_n(x)\rangle &= \langle \epsilon \mid Q^k p_n(x)\rangle \\ &= \langle \epsilon \mid (n)_k\, p_{n-k}(x)\rangle = n!\, \delta_{n,k}\,,\end{aligned}$$

so that $p_n(x)$ is the associated sequence for L.

The Expansion Theorem, stated in terms of shift-invariant operators, leads to another generalization of Taylor's formula:

PROPOSITION 7.6. *Let Q be a delta operator with associated sequence $p_n(x)$, and let T be a shift-invariant operator. Then*

$$T = \sum_{k=0}^{\infty} \frac{\langle \epsilon \mid Tp_k(x)\rangle}{k!} Q^k.$$

COROLLARY 1. *Let Q be a delta operator with associated sequence $p_n(x)$. Then*

$$E^y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} Q^k.$$

COROLLARY 2. *Let Q be a delta operator with associated sequence $p_n(x)$, then if $p(x)$ is any polynomial, we have*

$$p(x+y) = \sum_{k=0}^{\infty} \frac{Q^k p(x)}{k!} p_k(y).$$

For $Q = D$, Corollary 2 is precisely Taylor's formula.

If $Q = \mu(L)$ and $T = \mu(M)$, then the *Q-indicator of T* is the L-indicator of M.

We next consider automorphisms and derivations of the algebra Σ of shift-invariant operators. In view of the isomorphism $\mu: P^* \to \Sigma$ of Proposition 7.1, every automorphism γ of Σ is of the form $\gamma = \mu\beta\mu^{-1}$, for some automorphism β of P^*. In fact, every automorphism of Σ is related to a unique delta operator $Q = \mu(L)$ by $\beta_Q = \mu\beta_L\mu^{-1}$.

Similarly, every continuous derivation of Σ is of the form $\partial_Q = \mu\partial_L\mu^{-1}$. These characterizations can be made more explicit as follows:

THEOREM 7. (a) *Every continuous automorphism of the algebra of shift-invariant operators is of the form $T \to \alpha^{-1}T\alpha$, where α is an umbral operator, and conversely.*

(b) *Every continuous derivation of the algebra of shift-invariant operators is of the form $T \to T\theta - \theta T$, where θ is an umbral shift, and conversely.*

Proof. (a) Suppose β_Q is a continuous automorphism of Σ, where $Q = \mu(L)$. For a shift-invariant operator $T = \mu(M)$ we have

$$\beta_Q(T) = \mu\beta_L\mu^{-1}(T) = \mu\beta_L(M).$$

Now if N is a linear functional and $p(x)$ is a polynomial, we may write

$$\begin{aligned}\langle N \mid \mu(\beta_L(M))\, p(x)\rangle &= \langle \beta_L(M)\, N \mid p(x)\rangle \\ &= \langle \beta_L(M\beta_L^{-1}(N)) \mid p(x)\rangle = \langle M\beta_L^{-1}(N) \mid \beta_L{}^* p(x)\rangle \\ &= \langle \beta_L^{-1}(N) \mid \mu(M)\, \beta_L{}^* p(x)\rangle = \langle N \mid (\beta_L^{-1})^*\, T\beta_L{}^* p(x)\rangle.\end{aligned}$$

and thus $\beta_Q(T) = (\beta_L^{-1})^* T\beta_L^*$. The same argument proves the converse assertion.

(b) Let ∂_Q be an continuous derivation of Σ. If $Q = \mu(L)$ and if $T = \mu(M)$ is any shift-invariant operator, we have

$$\partial_Q(T) = \mu\partial_L\mu^{-1}(T) = \mu\partial_L(M).$$

If N is any linear functional, and $p(x)$ any polynomial, then

$$\begin{aligned}\langle N \mid \mu\partial_L(M)\,p(x)\rangle &= \langle \partial_L(M)N \mid p(x)\rangle \\ &= \langle \partial_L(MN) - M\partial_L N \mid p(x)\rangle \\ &= \langle N \mid (T\partial_L{}^* - \partial_L{}^* T)\,p(x)\rangle.\end{aligned}$$

Therefore, $\partial_Q(T) = T\partial_L{}^* - \partial_L{}^*T$. The converse is proved similarly.

As an application, we obtain a representation of umbral shifts.

THEOREM 8. *Let θ_L and θ_M be the umbral shifts associated with the delta functionals L and M, respectively, and let $Q = \mu(L)$ and $P = \mu(M)$. Then*

$$\theta_L = \theta_M(\partial_P Q)^{-1}.$$

Proof. By Proposition 6.5, $\partial_L M = (\partial_M L)^{-1}$ and so $\partial_L = (\partial_M L)^{-1}\,\partial_M$. Observing that $\partial_M L = \mu^{-1}\partial_P\mu(\mu^{-1}(Q)) = \mu^{-1}\partial_P Q$, for a linear functional N and polynomial $p(x)$, we have

$$\begin{aligned}\langle N \mid \theta_L\,p(x)\rangle &= \langle \partial_L N \mid p(x)\rangle \\ &= \langle (\partial_M L)^{-1}\,\partial_M N \mid p(x)\rangle = \langle \partial_M N \mid \mu[(\partial_M L)^{-1}]\,p(x)\rangle \\ &= \langle \partial_M N \mid [\mu(\partial_M L)]^{-1}\,p(x)\rangle \\ &= \langle \partial_M N \mid (\partial_P Q)^{-1}\,p(x)\rangle \\ &= \langle N \mid \theta_M(\partial_P Q)^{-1}\,p(x)\rangle.\end{aligned}$$

The conclusion follows.

By letting $M = A$ in the preceding theorem, we obtain

COROLLARY 1 (Recurrence Formula). *Let $p_n(x)$ be the associated sequence for the delta operator Q. Then*

$$p_{n+1}(x) = x(\partial_D Q)^{-1}\,p_n(x).$$

COROLLARY 2. *Let θ_Q be an umbral shift, with corresponding delta operator Q. Then*

$$Q\theta_Q - \theta_Q Q = I.$$

For the special case $Q = D$, the associated shift θ_D is the operator X of multiplication by x, and Corollary 2 reduces to the familiar formula $DX - XD = I$. For convenience we denote the operator $\partial_D T = TX - XT$ by T', and if $L = \mu^{-1}(T)$ we denote $\mu^{-1}(T')$ by L'.

As expected, the indicator of the operator $\partial_D Q$ is the derivative of the indicator of Q.

We conclude with some powerful formulas for computing the associated sequence for a delta operator.

THEOREM 9 (Transfer Formula). *If $Q = PD$ is a delta operator, where P is an invertible shift-invariant operator, and if $p_n(x)$ is the associated sequence for Q, then*

$$p_n(x) = Q'P^{-n-1}x^n$$

for all $n \geqslant 0$.

Proof. Letting $q_n(x) = Q'P^{-n-1}x^n$, we see that

$$Qq_n(x) = PDQ'P^{-n-1}x^n = nq_{n-1}(x)$$

and thus by Proposition 7.5 we need only show that $q_0(x) = 1$ and $q_n(0) = 0$ for $n > 0$.

It is clear that $q_0(x)$ is a constant. Furthermore,

$$\begin{aligned}\langle \epsilon \mid q_0(x)\rangle &= \langle \epsilon \mid Q'P^{-1}1\rangle = \langle \epsilon \mid (P + DP')\, P^{-1}1\rangle \\ &= \langle \epsilon \mid 1\rangle = 1,\end{aligned}$$

and we have $q_0(x) = 1$. For $n > 0$,

$$\begin{aligned}\langle \epsilon \mid q_n(x)\rangle &= \langle \epsilon \mid Q'p^{-n-1}x^n\rangle \\ &= \langle \epsilon \mid (P + DP')\, P^{-n-1}x^n\rangle \\ &= \langle \epsilon \mid P^{-n}x^n\rangle + \langle \epsilon \mid nP'P^{-n-1}x^{n-1}\rangle \\ &= \langle \epsilon \mid P^{-n}x^n\rangle - \langle \epsilon \mid (P^{-n})'x^{n-1}\rangle \\ &= \langle \epsilon \mid P^{-n}x^n\rangle - \langle \mu^{-1}(P^{-n}) \mid x^n\rangle \\ &= 0.\end{aligned}$$

Thus $q_n(0) = 0$ for $n > 0$ and the theorem is proved.

COROLLARY 1 (Transfer Formula). *If $Q = PD$ is a delta operator, with associated sequence $p_n(x)$, then*

$$p_n(x) = xP^{-n}x^{n-1}$$

for all $n \geqslant 1$.

Proof. The result follows from Theorem 9 and from the following computation:

$$\begin{aligned} Q'P^{-n-1}x^n &= (P + DP')\,P^{-n-1}x^n \\ &= P^{-n}x^n + nP'P^{-n-1}x^{n-1} \\ &= P^{-n}x^n - (P^{-n})'\,x^{n-1} \\ &= P^{-n}x^n - (P^{-n}X - XP^{-n})\,x^{n-1} \\ &= xP^{-n}x^{n-1}. \end{aligned}$$

COROLLARY 2. *Let Q be a delta operator, with associated sequence $p_n(x)$. Let $R = QT$ be another delta operator, with associated sequence $q_n(x)$, where T is an invertible shift invariant operator. Then*

$$q_n(x) = xT^{-n}x^{-1}p_n(x),$$

for $n \geqslant 1$.

Since any two delta operators Q and R are related by $QT = R$ for some invertible shift-invariant operator, Corollary 2 relates any two associated sequences.

8. EXAMPLES

We are now ready to show how the methods developed so far give an efficient technique for the computation of associated polynomials and connection constants. Specifically, to compute the matrix of constants $c_{n,k}$ in

$$p_n(x) = \sum_{k=0}^{n} c_{n,k}q_k(x),$$

where $p_n(x)$ and $q_n(x)$ are of binomial type, one uses the fact that the sequence $r_n(x) = \sum_k c_{n,k}x^k$ is also of binomial type, and that its indicator is computed by umbral methods in terms of the indicators for $p_n(x)$ and $q_n(x)$. Once the indicator for $r_n(x)$ is known, the coefficients of $r_n(x)$ are computed by one of the explicit formulas given in the previous section.

1.3. We have already remarked that the operator $\mu(A)$ is D, the ordinary derivative. Clearly $D' = I$, and the associated sequence is $p_n(x) = x^n$.

2.3. The *forward difference operator* is $\Delta_a = \mu(\epsilon_a - \epsilon) = E^a - I$, and its derivative is $\Delta_a' = aE^a$. To compute the associated sequence, we use the Recurrence Formula:

$$\begin{aligned} p_n(x) &= x(\Delta_a')^{-1}\,p_{n-1}(x) \\ &= xa^{-1}E^{-a}p_{n-1}(x) \\ &= a^{-1}xp_{n-1}(x-a), \end{aligned}$$

whence

$$\begin{aligned} p_n(x) &= a^{-n}x(x-a)(x-2a)\cdots(x-(n-1)a) \\ &= (x/a)_n , \end{aligned}$$

as previously announced.

We can use the Recurrence Formula to compute the conjugate sequence for $\epsilon_a - \epsilon = e^{aA} - \epsilon$ by computing the associated sequence for the conjugate functional $[\log(1+A)]/a$. Indeed, we have

$$\begin{aligned} q_n(x) &= xa(1+D)\, q_{n-1}(x) \\ &= \cdots = [ax(1+D)]^n 1 \\ &= e^{-x}(axD)^n\, e^x. \end{aligned}$$

We know from previous discussions that the $q_n(x)$ are the exponential polynomials. Thus we have proved the Stirling numbers identity

$$e^{-x}(xD)^n\, e^x = \sum_{k=0}^{n} S(n, k)\, x^k,$$

where $S(n, k)$ are the Stirling numbers of the second kind. It is easy to see by Rolle's theorem that these polynomials have real roots.

3.3. The *backward difference operator* $I - E^a$, with derivative aE^{-a}, is similarly treated, giving the associated polynomials $p_n(x) = \langle x/a\rangle_n$.

If Q is a delta operator with associated sequence $p_n(x)$, and if $q_n(x)$ is the associated sequence for the Abelization $R = QE^a$ of Q, then we have

$$\begin{aligned} q_n(x) &= xE^{-an}x^{-1}p_n(x) \\ &= \frac{x}{x-an}\, p_n(x-an). \end{aligned} \tag{$*$}$$

This specializes to a host of polynomial sequences studied in various circumstances.

4.3. The *Abel operator* $\mu(A\epsilon_a)$ is DE^a; hence its derivative is $(DE^a)' = E^a(1 + aD)$. The Transfer Formula computes the Abel polynomials

$$\begin{aligned} p_n(x) &= xE^{-an}x^{n-1} \\ &= x(x-an)^{n-1}. \end{aligned}$$

5.3. The *difference-Abel operator* is $E^a(E^b - I)$ and its derivative is $E^a((a + b)E^b - a)$. From Eq. (*), we compute the Gould polynomials

$$\begin{aligned} p_n(x) &= xE^{-an}x^{-1}(x/b)_n \\ &= \frac{x}{x - an}\left(\frac{x - an}{b}\right)_n . \end{aligned}$$

6.3. The *central difference operator* is $\mu(\delta_a) = E^{a/2} - E^{-a/2}$ and its derivative is $(E^{a/2} + E^{-a/2})/2$. For $a = 1$, Eq. (*) (with a replaced by $-a/2$) gives the Steffensen polynomials

$$\begin{aligned} p_n(x) &= xE^{n/2}x^{-1}(x)_n \\ &= x(x + n/2 - 1)_{n-1} = x^{[n]}. \end{aligned}$$

7.3. The *Laguerre operator* is $L = \mu(l) = D(D - I)^{-1}$. The Laguerre operator satisfies

$$Lp(x) = \int_{-\infty}^{0} e^t p'(x + t)\, dt.$$

To compute the derivative L', we recall that

$$L'p(x) = (LX - XL)\, p(x),$$

whence

$$L' = \int_{-\infty}^{0} te^t p'(x + t)\, dt.$$

Several expansions for the associated sequence can be obtained. By the Transfer Formula we have

$$L_n(x) = x(D - I)^n\, x^{n-1} = xe^x D^n e^{-x} x^{n-1},$$

which is the classical Rodriques formula. By the Transfer Formula,

$$\begin{aligned} L_n(x) &= L'(D - I)^{n+1}\, x^n \\ &= -(D - I)^{n-1}\, x^n \\ &= -e^x D^{n-1} e^{-x} x^n. \end{aligned}$$

Finally, expanding $(D - I)^{n-1}$ we obtain the coefficients explicitly:

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n - 1}{k - 1} (-x)^k.$$

Next we give some examples of computation of connection constants. By way of orientation, we repeat a classical instance:

• 2.4. Determine the constants $c_{n,k}$ in

$$(x)_n = \sum_{k=0}^{n} c_{n,k}\langle x\rangle_k .$$

Since $(x)_n$ is the associated sequence for $g(A) = e^A - \epsilon$ and $\langle x\rangle_n$ is the associated sequence for $f(A) = \epsilon - e^{-A}$, $g(f^{-1}(A)) = A/(\epsilon - A)$. Therefore, $r_n(x) = \sum_{k=0}^{n} c_{n,k}x^k = L_n(-x)$, where $L_n(x)$ are the (basic) Laguerre polynomials. One can hardly hope for anything simpler.

3.4. Determine the constants $c_{n,k}$ in

$$\langle x\rangle_n = \sum_{k=0}^{n} c_{n,k}\langle x/a\rangle_k .$$

Since $\langle x\rangle_n$ is the associated sequence for $g(A) = \epsilon - e^{-A}$, and $\langle x/a\rangle_n$ is the associated sequence for $f(A) = \epsilon - e^{-aA}$, we have $g(f^{-1}(A)) = \epsilon - (\epsilon - A)^a$. Thus by the Recurrence Formula,

$$\begin{aligned} r_n(x) &= xa(I - D)^{a-1}\, r_{n-1}(x) \\ &= \cdots = a^n(x(I - D)^{a-1})^n\, 1 \\ &= a^n e^x(xD)^n\, e^{-x}. \end{aligned}$$

• 4.4. Express the Abel polynomials as linear combinations of the Laguerre polynomials. That is, determine the constants $c_{n,k}$ such that

$$A_n(x, a) = \sum_{k=0}^{n} c_{n,k}L_k(x).$$

The sequence $L_n(x)$ is the associated sequence for $f(A) = A/(A - \epsilon)$, and $A_n(x, a)$ is the associated sequence for $g(A) = Ae^{aA}$. Thus $g(f^{-1}(A)) = [A/(A - 1)]\, e^{aA/(A-1)}$. By the Transfer Formula, the associated sequence for $g(f^{-1}(A))$ is

$$\begin{aligned} r_n(x) &= x(D - I)^n\, e^{-anD/(D-I)}x^{n-1} \\ &= xe^xD^ne^{-x}e^{-anD/(D-I)}x^{n-1}. \end{aligned}$$

The coefficients $c_{n,k}$ are now obtained by a routine Taylor expansion.

• 5.4. Determine the connection constants $c_{n,k}$ of the Gould polynomials with the factorial powers:

$$G_n(x, a, -1) = \sum_{k=0}^{n} c_{n,k}\langle x\rangle_k .$$

Again, $G_n(x, a, -1)$ is the associated sequence for $g(A) = e^{aA}(e^{-A} - \epsilon)$ and x_n is the associated sequence for $f(A) = \epsilon - e^{-A}$, hence, $g(f^{-1}(A)) = -A(\epsilon - A)^{-a}$ By the Transfer Formula

$$\begin{aligned} r_n(x) &= (-1)^n \, x(I - D)^{an} \, x^{n-1} \\ &= (-1)^n \sum_{k \geqslant 0} \binom{an}{k} (-1)^k \, (n-1)_k \, x^{n-k}, \end{aligned}$$

a relative of the Laguerre sequence.

• 6.4. Determine the connection constants $c_{n,k}$ of the Steffensen polynomials with the factorial powers:

$$x^{[n]} = \sum_{k=0}^{n} c_{n,k}(x)_k \, . \quad \checkmark$$

The Steffensen polynomials are the associated polynomials for $\delta = \epsilon_{-1/2}(\epsilon_1 - \epsilon)$. In this case $g(A) = e^{-A/2}(e^A - \epsilon)$ and $f(A) = e^A - \epsilon$ Thus $g(f^{-1}(A)) = A(\epsilon + A)^{-1/2}$ and

$$\begin{aligned} r_n(x) &= x(1 + D)^{n/2} \, x^{n-1} \\ &= \sum_{k \geqslant 0} \binom{n/2}{k} (n-1)_k \, x^{n-k}, \end{aligned}$$

again a most explicit answer.

7 4. We derive Erdelyi's duplication formulas for Laguerre polynomials; that is, we determine the $c_{n,k}$ for which

$$L_n(ax) = \sum_{k=0}^{n} c_{n,k} L_k(x).$$

Now $L_n(ax)$ is the associated sequence for $g(A) = a^{-1}A/(a^{-1}A - \epsilon)$ and therefore $r_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$ is the associated sequence for $A/[(\epsilon - a)A + a]$. By the Transfer Formula,

$$\begin{aligned} r_n(x) &= x((1 - a) \, D + aI)^n \, x^{n-1} \\ &= \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (1 - a)^{n-k} \, (ax)^k. \end{aligned}$$

Thus:

$$L_n(ax) = \sum_{k \geqslant 0} \frac{n!}{k!} \binom{n-1}{k-1} (1 - a)^{n-k} \, a^k L_k(x).$$

2.5. We give some applications of umbral techniques to the Stirling numbers $s(n, k)$ and $S(n, k)$ of the first and second kind. Recall that the exponential polynomials

$$\phi_n(x) = \sum_{k=0}^{n} S(n, k)\, x^k$$

are the associated polynomials for the delta functional $\log(\epsilon + A)$.

The Recurrence Formula gives a recurrence formula for the exponential polynomials:

$$\begin{aligned}\phi_n(x) &= x(I + D)\,\phi_{n-1}(x)\\ &= x(\phi_{n-1}(x) + \phi'_{n-1}(x)).\end{aligned}$$

Dobinsky's formula is practically trivial. Letting $p_n(x) = (x)_n$, we take an umbral composition

$$p_n(\phi(x)) = x^n = e^{-x} \sum_{k\geqslant 0} \frac{p_n(k)}{k!}\, x^k,$$

and thus for any polynomial $p(x)$,

$$p(\phi(x)) = e^{-x} \sum_{k\geqslant 0} \frac{p(k)}{k!}\, x^k.$$

For $p(x) = x^n$, we obtain Dobinsky's formula:

$$\phi_n(x) = e^{-x} \sum_{k\geqslant 0} \frac{k^n x^n}{k!}.$$

Consider the polynomials

$$\psi_n(x) = \sum_{k=0}^{n} s(n, k)(x)_k .$$

If we define the umbral operators $\alpha: x^n \to (x)_n$ and $\beta: x^n \to \phi_n(x)$, then Corollary 1 to Theorem 5 gives $\alpha^{-1} = \beta$. Therefore,

$$\psi_n(\phi(x)) = \beta(\psi_n(x)) = \beta\alpha(x)_n = (x)_n ,$$

or, more explicitly:

$$s(n, k) = \sum_{i,j\geqslant 0} s(n, j)\, s(j, i)\, S(i, k).$$

Similarly, from $\phi_n(\psi(x)) = (x)_n$, we obtain

$$\sum_{k\geqslant 0} S(n, k)\, s(k, i) = \delta_{n,i} .$$

We can derive another recurrence for the exponential polynomials as follows. If α is an umbral operator, then

$$\alpha^*(A^k)' = \alpha^*(A^{k'})\,\alpha^*(A)'.$$

Applying to a polynomial $p(x)$ and using the properties of adjoints and derivations:

$$\langle A^k \mid \alpha x p(x)\rangle = \langle A^k \mid x\alpha(\mu(\alpha^*(A)')\,p(x))\rangle.$$

Therefore,

$$\alpha x p(x) = x\alpha(\mu(\alpha^*(A)')\,p(x)).$$

Now if we take $\alpha: x^n \to \phi_n(x)$, then $\alpha^*(A) = e^A - \epsilon$ and so $\mu(\alpha^*(A)') = E^1$. Setting $p(x) = x^n$ gives

$$\phi_{n+1}(x) = x(\phi + 1)^n,$$

which, in terms of coefficients, gives the Stirling numbers recurrence

$$S(n+1, k) = \sum_{i \geqslant 0} \binom{n}{i} S(i, k-1).$$

9. Sheffer Sequences

So far, we have no explicit formula for shift-invariant operators. In obtaining an explicit formula for $\mu(L)$, we are led to a new class of polynomial sequences. A polynomial sequence $s_n(x)$ is a *Sheffer sequence relative to* a sequence $p_n(x)$ of binomial type if it satisfies the functional equation

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, p_{n-k}(y)$$

for all $n \geqslant 0$ and for all $y \in K$.

Some characterizations of Sheffer sequences follow. The proofs follow a familiar pattern, and are therefore omitted.

Proposition 9.1. *A polynomial sequence $s_n(x)$ is a Sheffer sequence if and only if there exist a sequence of binomial type $p_n(x)$ and an invertible shift-invariant operator P such that*

$$p_n(x) = P s_n(x)$$

for all $n \geqslant 0$.