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The Umbral Calculus

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DEDICATED TO THE MEMORY OF NORMAN LEVINSON

Contents.

1. INTRODUCTION

There are three known ways of describing a sequence of numbers a_0, a_1, a_2, a_3,...:

(1) By recursion. Here, a specific rule f is given whereby $a_n = f(a_{n-1}, a_{n-2}, \ldots)$. This description is used whenever the sequence is to be explicitly computed.

(2) By generating functions. Here, the description of the sequence is thrown back on that of the function

$$f(x) = a_0 + a_1x + a_2x^2/2! + \cdots.$$

This description has proved effective when the asymptotic properties of the sequence are sought.

(3) By transform methods. Here, the sequence is represented as the result of performing a definite integral, for example as a moment sequence, say

$$a_n = \int_0^1 x^n f(x)\, dx,$$

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and the properties of a_n are thrown back on "corresponding" properties of the function $f(x)$. Stripped of irrelevancies, this method reduces to representing the sequence a_n as the result of applying a linear functional L to the sequence of polynomials x^n. Adopting the physicists' notation, we write this action as $\langle L \mid x^n \rangle = a_n$.

In the nineteenth century—and among combinatorialists well into the twentieth—the linear functional L would be called an umbra, a term coined by Sylvester, that great inventor of unsuccessful terminology. Before knowledge of linear algebra became widespread, the action of a linear functional L would be conceived of as raising the index n to a power, and then "treating" the sequence a_n as a sequence of powers a^n, while reserving the right to lower the index at the proper time. No precise rules for lowering of indices were stated, nor could they be, as long as the underlying conceptual framework was missing. A baffling difficulty in the calculus of umbrae was the important "rule"

$$(a + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k a^{n-k},$$

which seemed to imply $a + a \neq 2a$.

If mathematicians had held back their tendency to disregard techniques, even though useful, that do not conform to the standards of rigor of the day, they might have been led, by an analysis of umbrae, to the concept of Hopf algebra. Unfortunately, this was a missed opportunity, and the concept was to emerge much later from algebraic topology. Briefly, it was recognized that linear functionals on polynomials can not only be added, but also multiplied according to the rule

$$\langle L_1 L_2 \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_1 \mid x^k \rangle \langle L_2 \mid x^{n-k} \rangle.$$

The resulting pairing of two rings leads to a powerful formalism, which it is the purpose of this work to develop.

A vast variety of special polynomial sequences occurs in combinatorics and in analysis. It was recognized in a previous work that these sequences of polynomials $p_n(x)$, which we have called of *binomial type*, satisfy the identity

$$p_n(x + a) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)\, p_{n-k}(x).$$

We show that these sequences can be defined by a simple orthogonalization device. They are related to a linear functional L such that $\langle L \mid 1 \rangle = 0$ by the biorthogonality conditions

$$\langle L^k \mid p_n(x) \rangle = n!\, \delta_{n,k}.$$

We develop the theory of sequences of binomial type starting from this condition. From the point of view of computation, the two most important problems are, first, that of effectively calculating a sequence of binomial type once L is given, and second, that of computing the *connection constants* $c_{n,k}$ between two sequences of binomial type $p_n(x)$ and $q_n(x)$:

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} p_k(x).$$

Our solution consists in describing the polynomial sequence $r_n(x) = \sum_k c_{n,k} x^k$ as a sequence of binomial type whose functional L is computed in a very simple way in terms of those of $p_n(x)$ and $q_n(x)$.

Polynomial sequences of binomial type turn out in large variety in problems of enumeration. Roughly speaking, problems of enumerating objects that are pieced together out of smaller objects which are not allowed to overlap—for example, the enumeration of trees—fall within the scope of the present theory. A sprinkling of examples given at the end is meant to foreshadow a more substantial development in this direction which we were forced to postpone.

The text has been supplemented by several examples from analysis which have occurred in various circumstances, mostly in connection with expansion of functions into series of polynomials, such as Taylor's, Newton's or Euler–MacLaurin's. That such expansions, as well as sundry other properties of special polynomial sequences, turn out to be special cases of a few exceedingly simple facts, is not only a pleasing realization. It is hoped that it will encourage the use of the simple general techniques of the umbral calculus, and discourage the collector's mentality that considers each polynomial sequence as an inviolable manifestation of a unique phenomenon.

Among the by-products of the present theory is an effective formalism for computation involving composition with formal power series and Lagrange inversion. A great deal of combinatorics depends on these computations, and the classical notation of the calculus offers little relief. A linear functional L on polynomials such that $\langle L \mid x^n \rangle = a_n$ corresponds to the formal power series whose nth coefficient is a_n, and this algebraic isomorphism leads to a swift technique for functional composition and inversion, as can be gleaned from the examples in Section 11.

Paradoxically, this identification of linear functionals with formal power series is one reason why a development along the present lines was overlooked. But it would be just as arbitrary to identify linear functionals with distributions, or with some yet-to-be-conceived gadget. The simplifying power of the present notation occurs out of the ease of handling adjoints of linear operators in the vector space duality between polynomials and functionals, and would be lost, had functionals been identified from the start with formal power series.

Another by-product of the present work is the theory of factor sequences, which allows for "polynomials" of negative degree, and which can be considered as an extension of the theory of factorial series to arbitrary sequences of binomial type. Thus we can define Hermite, Bernoulli, Stirling polynomials, etc., of negative degree. Whereas the generating functions of sequences of polynomials of binomial type, as well as the closely related Sheffer sequences, are expressed by exponentials, their analogs for factor sequences lead us to define an "integral" analog of the notion of formal power series, which we propose to call the Cigler transform, as it partially answers a question posed by J. Cigler.

Throughout, some definitions and elementary results could have been presented as special cases of Hopf algebra notions, but we have avoided this line, partly because Hopf algebras are still little known, and partly because it is left as a challenge to Hopf algebraists to generalize some of our notions, for example, factor sequences, the adjointness between shifts and derivations, and umbral composition, to their rarefied atmosphere.

A great many of the results in this work are new. Others are taken from our previous work on this subject. In the choice of examples, we have preferred to rely on established polynomial sequences rather than describe new sequences which could not be properly motivated. Altogether, this work may be compared to the archeologist's putting together of a dinosaur out of a few charred bones in the desert.

2. Survey

The notion of polynomial sequence of binomial type goes back to E. T. Bell and probably earlier. Steffensen was the first to observe that sequences associated with delta operators in the same way as D is to x^n are of binomial type, but failed to notice the converse of this fact, which was first stated and proved by Mullin and Rota.

The idea of associated and conjugate polynomials is first developed here. The history of the subject has been sketched in "Finite Operator Calculus."

The isomorphism between the umbral algebra and the algebra of shift-invariant operators, first seen by the Hopf algebraists, has not yet made much headway elsewhere. Thus Feller in his treatise on probability dedicates two separate chapters to Fourier transforms and to convolution operators, and correspondingly gives two proofs of the Central Limit Theorem, little realizing that they are really one and the same proof. The use of linear functionals and of the augmentation—that is, evaluation at zero—in place of operators results in substantial simplifications; computations of composition and inversion of power series become transparent in terms of the duality between the algebra of polynomials and the umbral algebra of linear functionals (Sections 6 and 11); the Lagrange inversion formula, for example, boils down to the computation of the adjoint of an operator.

It remains a mystery why so many polynomial sequences occurring in various mathematical circumstances turn out to be of binomial type. The explanation we give in terms of automorphisms of the umbral algebra can be recast in terms of the Weyl algebra in one pair of generators, that is, the associative algebra freely generated by two variables P and Q subject to the identity $PQ - QP = I$. Every sequence of binomial type determines a module over the Weyl algebra, and such modules are easily characterized.

The Weyl algebra approach is followed by J. Cigler in the study of factor sequences—the name is ours—but at considerable expense: in Cigler's approach the analog of Proposition 10.2 fails and as a consequence the computation of associated factor sequences becomes difficult and sometimes impossible to state.

The theory of factor sequences is barely scratched here, and it suggests the reopening of a number of questions in the calculus of finite differences which have lain dormant since Nörlund and Pincherle. The analogy between differential and difference equations, long considered a baffling coincidence, can now be seen as a special case of a theory of Q-difference equations, when Q is an arbitrary delta operator, each Q leading to its own theory of isolated singularities much as in the case of linear differential equations with rational coefficients. The purely algebraic connection between factor sequences and formal power series (Section 11) may be useful in developing a purely algebraic theory of singularities of Q-difference equations. For example, the analogy between $\log x$, "the" solution of $Dy = 1/x$, and $\psi(x)$, "the" solution of $\Delta y = 1/x$, leads more generally to the study of the Q-equations $Qy = 1/x$. Similarly, R. M. Cohn's difference algebra, conceived as a difference analog of Ritt's Galois theory for differential equations, is a good candidate for extension to delta operators.

Again, the combined use of polynomials and factor sequences does away with notions of convergence, or even of asymptotic approximation. It seems furthermore that the notion of "formal" definite integral, which we propose to call the Cigler transform, relates to those asymptotic expansions which arise from stationary phase.

We cannot pass under silence a conceptual problem arising from factor sequences. Every sequence of binomial type is the sequence of eigenfunctions of the operator $\theta_L \mu(L)$ in a suitable Hilbert space. What, then, is the spectral nature of those "eigenfunctions of negative order" that are the associated factor sequence? Does this phenomenon call for a retouching of the notion of eigenfunction expansion? Hilbert space considerations could also be called in to give, with the aid of the adjointness between umbral operators and automorphisms (Section 6), a simple solution of the problem of conjugacy of formal power series which has a good chance of extending to the multivariate case.

The sprinkling of examples is not meant to be exhaustive, and we were forced to defer some applications of umbral techniques, such as: a general understanding of Turán-type inequalities by sums of sequences (we give two examples), a goal toward which Al-Salam, Carlitz, Toscano, and others have contributed some dazzling spade work; a theory of "formal" partial fraction expansions; and a structural study of the Laguerre polynomials. These polynomials play a role in far too many questions, and their formal analogies with Hermite polynomials have not been satisfactorily explained. One can, for example, develop Feynman diagram representations of integrals of products of Laguerre polynomials, in analogy with Hermite. Does this mean that the Laguerre polynomials are associated with a yet-to-be-discovered stochastic process, as Hermite polynomials are to Brownian motion?

The combinatorial examples given in Section 14 are meant only as hints. A more systematic correspondence between operations on polynomial sequences of binomial type and set-theoretic operation on partitionals can and will be presented elsewhere. For example, umbral composition corresponds to a set-theoretic "composition" of two stores. Polynomial sequences with alternating, though still integer, coefficients can be interpreted by a sieve that expresses one store as resulting from the composition of two stores.

There is, however, a more promising set-theoretic interpretation of polynomial sequences of binomial type. Let $\mathbf{B}$ be a ring of subsets of a set S, that is, a family of subsets closed under unions, intersections, and relative complements. The *Poisson algebra* of $\mathbf{B}$ is the Boolean algebra $p(\mathbf{B})$ generated by elements denoted by (A, n), where $A \in \mathbf{B}$ and n is a nonnegative integer, subject to the identities $(A \cap B, n) = \bigcup_{i=0} ((A, i) \cap (B, n - i))$ for A and B disjoint, and $(A, n)^c = \bigcup_{i \neq n} (A, i)$. If μ is a measure on B, a signed measure π on $p(\mathbf{B})$ is said to be μ-invariant when $\pi((A, n)) = \pi((B, n))$ if $\mu(A) = \mu(B)$. It can be shown—subject to mild restrictions—that every μ-invariant measure on a Poisson algebra $p(\mathbf{B})$ is of the form $\pi((A, n)) = p_n(\mu(A)) \exp(\lambda\mu(A))$, when λ is a constant and $p_n(x)$ is a sequence of polynomials of binomial type. On the basis of this result, the umbral calculus can be systematically interpreted as a calculus of measures on Poisson algebras, generalizing compound Poisson processes. This interpretation in turn suggests a vast generalization of the umbral calculus, corresponding to measures on a Poisson algebra that are not assumed μ-invariant.

In addition to reiterating the acknowledgments given in "Finite Operator Calculus," we wish to express our indebtedness to the work of J. Cigler, A. Garsia, and especially J. Delsarte, whose pioneering contributions we have unpardonably failed to mention in previous works.

3. The Umbral Algebra

Let P denote the commutative algebra of all polynomials in a single variable x, with coefficients in a field K of characteristic zero, which we often assume to be either the real or the complex field. Let P^* be the vector space of all linear functionals on P. We denote the action of a linear functional L on a polynomial $p(x)$ by

$$\langle L \mid p(x)\rangle.$$

A *polynomial sequence* $p_n(x)$, $n = 0, 1, 2,\ldots$, is a sequence of polynomials where $p_n(x)$ is of degree n. It is clear that two linear functionals L and M are equal if and only if

$$\langle L \mid p_n(x)\rangle = \langle M \mid p_n(x)\rangle$$

for all $p_n(x)$ in a polynomial sequence. We will frequently use this argument, which we call *the spanning argument*. By the spanning argument, a linear functional L is defined once $\langle L \mid p_n(x)\rangle$ is given for all $p_n(x)$ in a polynomial sequence.

We make the vector space P^* into an algebra by defining the *product* of two linear functionals L and M by

$$\langle LM \mid x^n\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid x^k\rangle\langle M \mid x^{n-k}\rangle.$$

It is straightforward to verify

Proposition 3.1. *The product of linear functionals is commutative and associative.*

For a constant a, the linear functional ϵ_a, defined by

$$\langle \epsilon_a \mid p(x)\rangle = p(a),$$

is called *evaluation* at a. We write ϵ in place of ϵ_0, and call this linear functional the *augmentation*. It is easy to see that $\epsilon_a\epsilon_b = \epsilon_{a+b}$. Furthermore,

Proposition 3.2. *The augmentation is an identity for the product defined above.*

Thus the vector space of linear functionals P^*, with the above product and identity, is an algebra, which will be called the *umbral algebra*.

The umbral algebra is related to the algebra of functions of a real variable under convolution. Let f and g be functions with the property that

$$\int_{-\infty}^{\infty} f(x)\, x^n\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} g(x)\, x^n\, dx$$

are defined for all integers $n \geqslant 0$. Define linear functionals L_f and L_g by

$$\langle L_f \mid p(x)\rangle = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx,$$

$$\langle L_g \mid p(x)\rangle = \int_{-\infty}^{\infty} g(x)\, p(x)\, dx;$$

then the product $L_f L_g$ is the linear functional

$$\langle L_f L_g \mid p(x)\rangle = \int_{-\infty}^{\infty} h(x)\, p(x)\, dx,$$

where the function $h(x)$ is the convolution of the functions $f(x)$ and $g(x)$:

$$h(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

A major portion of the sequel is concerned with the study of a special class of polynomial sequences. A polynomial sequence $p_n(x)$ is said to be of ***binomial type*** if $p_0(x) = 1$ and if it satisfies the ***binomial identity***,

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y)$$

for all n, x, and y. For example, the sequence $p_n(x) = x^n$ is of binomial type.

The binomial identity yields, by iteration, the ***multinomial identity***:

$$p_n(x_1 + x_2 + \cdots + x_k) = \sum \binom{n}{j_1, \ldots, j_k} p_{j_1}(x_1) \cdots p_{j_k}(x_k),$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, \ldots, j_k)$ for which $j_1 + \cdots + j_k = n$.

The product of two linear functionals can be computed by using any sequences of binomial type in place of the sequence x^n. In particular,

PROPOSITION 3.3. *If $p_n(x)$ is a sequence of binomial type and if L and M are linear functionals, then*

$$\langle LM \mid p_n(x)\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle \langle M \mid p_{n-k}(x)\rangle.$$

Proof. Let $P[x, y]$ be the vector space of polynomials in the variables x and y. A linear functional L on P defines a linear operator L_x on $P[x, y]$ as follows. If $p(x, y) = \sum_{i,j} a_{i,j} x^i y^j$, then

$$L_x p(x, y) = \sum_{i,j} a_{i,j} \langle L \mid x^i\rangle\, y^j.$$

Similarly, the linear operator L_y is defined by

$$L_y p(x, y) = \sum_{i,j} a_{i,j} x^i \langle L \mid x^j \rangle.$$

In this notation the identity defining the product of two linear functionals L and M becomes

$$\langle LM \mid x^n \rangle = L_x M_y (x + y)^n.$$

By the spanning argument, the same identity holds for any polynomial $p(x)$:

$$\langle LM \mid p(x) \rangle = L_x M_y p(x + y).$$

The conclusion now follows by setting $p(x) = p_n(x)$ and expanding the right side by the binomial identity.

One proves similarly, using the multinomial identity,

PROPOSITION 3.4. *If $p_n(x)$ is a sequence of binomial type, and if $L_1, L_2, ..., L_k$ are linear functionals, then*

$$\langle L_1 L_2 \cdots L_k \mid p_n(x) \rangle = \sum \binom{n}{j_1, ..., j_k} \langle L_1 \mid p_{j_1}(x) \rangle \cdots \langle L_k \mid p_{j_k}(x) \rangle, \qquad (*)$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, ..., j_k)$ for which $j_1 + \cdots + j_k = n$.

One of the key properties of the product of linear functionals is

PROPOSITION 3.5. *Let L be a linear functional such that $\langle L \mid 1 \rangle = \langle L \mid x \rangle = \cdots = \langle L \mid x^{m-1} \rangle = 0$. Then*

$$\langle L^k \mid x^n \rangle = 0 \qquad \textit{for} \quad n < km.$$

Moreover,

$$\langle L^k \mid x^{km} \rangle = \frac{(km)!}{(m!)^k} \langle L \mid x^m \rangle^k.$$

Proof. Identity $(*)$, with $p_n(x) = x^n$ and $L_i = L$ for all $i = 1, 2, ..., k$ gives

$$\langle L^k \mid x^n \rangle = \sum \binom{n}{j_1, ..., j_k} \langle L \mid x^{j_1} \rangle \cdots \langle L \mid x^{j_k} \rangle,$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, ..., j_k)$ with $j_1 + \cdots + j_k = n$. If $n < km$, then $j_1 + \cdots + j_k < km$ and each term in the identity has a factor of the form $\langle L \mid x^{j_i} \rangle$ with $j_i < m$, and therefore equals zero. This establishes the first assertion.

When $n = km$, the only possible nonzero term in the identity comes when $j_i = m$ for all $i = 1, 2, \ldots, k$. The second assertion follows.

A frequently used special case of the preceding proposition is

COROLLARY 1. *If L is a linear functional such that $\langle L \mid 1 \rangle = 0$, then $\langle L^k \mid p(x) \rangle = 0$ for $k > \deg p(x)$.*

The umbral algebra P^* is a topological algebra under the topology defined as follows. A sequence L_n of linear functionals converges to a linear functional L whenever, given a polynomial $p(x)$, there exists an index n_0, depending on $p(x)$, such that for all $n > n_0$,

$$\langle L_n \mid p(x) \rangle = \langle L \mid p(x) \rangle.$$

Equivalently, an infinite series $\sum_{n \geqslant 0} L_n$ of linear functionals converges if and only if, given a polynomial $p(x)$, there is an index n_0 such that, for $n > n_0$,

$$\langle L_n \mid p(x) \rangle = 0.$$

In other words, the series $\sum_{n \geqslant 0} L_n$ converges if and only if the sequence L_n converges to zero. Under this topology, P^* becomes a complete topological algebra.

PROPOSITION 3.6. *For a linear functional L, and for a sequence of constants a_k, the following are equivalent:*

(i) $\langle L \mid 1 \rangle = 0$,

(ii) *the sequence L^k converges to zero,*

(iii) *the series $\sum_{k=0}^{\infty} a_k L^k$ converges.*

Proof. The equivalence of (ii) and (iii) follows from the definition of convergence, as remarked above. To see that (i) and (ii) are equivalent, notice that, if $\langle L \mid 1 \rangle = 0$, then by Corollary 1 to Proposition 3.5, $\langle L^k \mid p(x) \rangle = 0$ whenever $k > \deg p(x)$. Thus L^k converges to zero. Conversely, if $\langle L \mid 1 \rangle \neq 0$, then $\langle L^n \mid 1 \rangle = \langle L \mid 1 \rangle^n \neq 0$ for all $n \geqslant 0$, and so L^n cannot converge to zero.

In the sequel, the umbral algebra is always understood as a topological algebra.

As a final remark, if $p_n(x)$ is a polynomial sequence, then a sequence of linear functionals L_k such that

$$\langle L_k \mid p_n(x) \rangle = \delta_{k,n}$$

is not a basis for the vector space P^*, but only a pseudobasis. That is, every linear functional L can be uniquely expressed as a convergent series

$$L = \sum_{k=0}^{\infty} a_k L_k ,$$

where $a_k = \langle L \mid p_k(x) \rangle$. For the condition $\langle L_k \mid p_n(x) \rangle = \delta_{k,n}$ assures convergence of the series and its convergence to L follows by the spanning argument.

4. Delta Functionals

A *delta functional* is a linear functional L with the property that $\langle L \mid 1 \rangle = 0$ and $\langle L \mid x \rangle \neq 0$.

In this section we establish four main results. We show that to every delta functional one can associate two sequences of polynomials of binomial type. Using one of the sequences, we generalize Taylor's formula. We also establish an isomorphism between the topological algebra of linear functionals and the algebra of formal power series.

We begin by examining a classical special case. Consider the delta functional A, called the *generator*, defined by $\langle A \mid x^n \rangle = \delta_{n,1}$. For the sequence $p_n(x) = x^n$, Proposition 3.4 gives $\langle A^k \mid x^n \rangle = n!\, \delta_{n,k}$. In other words, the sequence x^n and the powers of the linear functional A form a *biorthogonal set.* This idea of biorthogonality will now be generalized.

A polynomial sequence $p_n(x)$ is the *associated sequence* for a delta functional L when

$$\langle L^k \mid p_n(x) \rangle = n!\, \delta_{n,k} \tag{$*$}$$

for all integers n, $k \geqslant 0$ (we set $L^0 = \epsilon$). Proposition 3.4 gives

Lemma 1. *If $p_n(x)$ is a sequence of binomial type and L is a delta functional then*

$$\langle L^n \mid p_n(x) \rangle = n! \langle L \mid p_1(x) \rangle^n \neq 0.$$

This allows us to prove

Proposition 4.1. *Every delta functional has a unique associated sequence.*

Proof. Let $p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ be a polynomial sequence. We show that $(*)$ uniquely defines the coefficients $a_{n,k}$. For $k = n$, $(*)$ gives

$$n! = a_{n,n} \langle L^n \mid x^n \rangle$$

and in view of the lemma, this uniquely defines $a_{n,n}$. We now proceed by induction. Assuming that $a_{n,n}$, $a_{n,n-1}$,..., $a_{n,n-i}$ have been defined, we show the same is true for $a_{n,n-i-1}$. By Proposition 3.5,

$$\left\langle L^{n-i-1} \,\Big|\, \sum_{k=0}^{n} a_{n,k} x^k \right\rangle = \sum_{k=n-i-1}^{n} a_{n,k} \langle L^{n-i-1} \mid x^k \rangle,$$

and this, together with (∗), yields

$$a_{n,n-i-1} \langle L^{n-i-1} \mid x^{n-i-1} \rangle = n!\, \delta_{n,n-i-1} - \sum_{k=n-i}^{n} a_{n,k} \langle L^{n-i-1} \mid x^k \rangle.$$

Since $\langle L^{n-i-1} \mid x^{n-i-1} \rangle \neq 0$, $a_{n,n-i-1}$ is uniquely defined. Q.E.D.

Since $L^0 = \epsilon$, (∗) implies that $p_0(x) = 1$ and $p_n(0) = 0$ for $n > 0$.

We wish to show that the associated sequence for a delta functional is a sequence of binomial type, and conversely. To this end, we derive the following generalization of Taylor's expansion:

THEOREM 1 (Expansion Theorem). *Let M be a linear functional and let L be a delta functional with associated sequence $p_n(x)$. Then*

$$M = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!} L^k.$$

Proof. The result follows from the spanning argument, noting that

$$\sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!} \langle L^k \mid p_n(x) \rangle = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!} n!\, \delta_{n,k} = \langle M \mid p_n(x) \rangle.$$

The following uniqueness assertion is implicit in the preceding proof.

COROLLARY 1. *Let M be a linear functional and let L be a delta functional. Suppose that*

$$M = \sum_{k=0}^{\infty} a_k L^k$$

for a_k in K. Then $a_k = \langle M \mid p_k(x) \rangle / k!$, where $p_k(x)$ is the associated sequence for L.

The Expansion Theorem says that every linear functional is in the closure of the linear span of the sequence of powers of a delta functional L. Thus, if $\langle L^k \mid p(x) \rangle = 0$ for all $k \geqslant 0$, we have $\langle M \mid p(x) \rangle = 0$ for all linear functionals M. This implies that $p(x) = 0$. We will use this argument many times in the sequel.

We come now to a main result:

THEOREM 2. (a) *Every associated sequence is a sequence of binomial type.*

(b) *Every sequence of binomial type is an associated sequence.*

Proof. (a) Let $p_n(x)$ be the associated sequence for the delta functional L. For nonnegative integers i and j, the definition of associated sequence gives

$$\langle L^i L^j \mid p_n(x)\rangle = \sum_{k=0}^{\infty} \binom{n}{k} \langle L^i \mid p_k(x)\rangle\langle L^j \mid p_{n-k}(x)\rangle. \tag{**}$$

Now if M and N are linear functionals with expansions

$$M = \sum_{k=0}^{\infty} a_k L^k$$

and

$$N = \sum_{k=0}^{\infty} b_k L^k,$$

a continuity argument together with (**) implies

$$\begin{aligned}\langle MN \mid p_n(x)\rangle &= \left\langle \sum_i a_i L^i \sum_j b_j L^j \mid p_n(x)\right\rangle \\ &= \sum_{i,j} a_i b_j \langle L^i L^j \mid p_n(x)\rangle \\ &= \sum_{i,j} a_i b_j \sum_{k=0}^{n} \binom{n}{k} \langle L^i \mid p_k(x)\rangle\langle L^j \mid p_{n-k}(x)\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \left\langle \sum_i a_i L^i \mid p_k(x)\right\rangle \left\langle \sum_j b_j L^j \mid p_{n-k}(x)\right\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle M \mid p_k(x)\rangle\langle N \mid p_{n-k}(x)\rangle.\end{aligned}$$

Letting $M = \epsilon_a$, $N = \epsilon_b$ and recalling that $\epsilon_a \epsilon_b = \epsilon_{a+b}$, we conclude that

$$p_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)\, p_{n-k}(b)$$

for all a and b, as desired.

(b) Let $p_n(x)$ be a sequence of binomial type. Define a sequence of linear functionals L_k by the biorthogonality conditions

$$\langle L_k \mid p_n(x)\rangle = n!\, \delta_{n,k}.$$

In particular, $\langle L_1 \mid 1\rangle = 0$ and $\langle L_1 \mid p_1(x)\rangle \neq 0$. Thus L_1 is a delta functional. The proof will be complete if we show that $L_i = L_1{}^i$, for $i \geqslant 0$, or equivalently, that $L_iL_j = L_{i+j}$ for $i, j \geqslant 0$. But this follows from the spanning argument since

$$\langle L_iL_j \mid p_n(x)\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_i \mid p_k(x)\rangle\langle L_j \mid p_{n-k}(x)\rangle$$

$$= \sum_{k=0}^{n} \binom{n}{k} k!\, \delta_{i,k}(n-k)!\, \delta_{j,n-k} = n!\, \delta_{n,i+j} = \langle L_{i+j} \mid p_n(x)\rangle$$

for all $n \geqslant 0$. Thus part (b) is proved.

Our first goal has been achieved, and we turn to further corollaries of the expansion theorem.

COROLLARY 2. *Let M and N be linear functionals, and let L be a delta functional. Suppose*

$$M = \sum_{k=0}^{\infty} a_k L^k, \qquad a_k \in K,$$

and

$$N = \sum_{k=0}^{\infty} b_k L^k, \qquad b_k \in K.$$

Then if

$$MN = \sum_{k=0}^{\infty} c_k L^k, \qquad c_k \in K,$$

we have

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}\,.$$

The preceding corollary leads to a simple criterion for invertibility of a linear functional:

COROLLARY 3. *A linear functional M is invertible in the umbral algebra if and only if $\langle M \mid 1\rangle \neq 0$.*

Proof. In the notation of the preceding corollary, if $a_0 = \langle M \mid 1\rangle \neq 0$, then setting $c_0 = 1$ and $c_k = 0$ for $k \geqslant 1$ we may solve successively for the coefficients b_k, and thereby determine the series expansion for a linear functional N, which is inverse to M. Conversely, if $\langle M \mid 1\rangle = 0$, then M is not invertible since it has a nontrivial null space.

Setting $M = \epsilon_y$ in the Expansion Theorem, we find

COROLLARY 4. *If L is a delta functional with associated sequence $p_n(x)$, then*

$$\epsilon_y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} L^k.$$

Any polynomial is a linear combination of a finite number of $p_n(x)$. The coefficients of such a linear combination are given by

COROLLARY 5. *If $p_n(x)$ is the associated sequence for the delta functional L, and if $p(x)$ is a polynomial, then*

$$p(x) = \sum_{k \geqslant 0} \frac{\langle L^k \mid p(x) \rangle}{k!} p_k(x).$$

By Corollary 1 to Proposition 3.5, all but a finite number of terms in the above sum are zero.

We proceed now to the next main result. By virtue of the Expansion Theorem, given a delta functional L we may associate to every linear functional M a formal power series in a single variable. In fact, if

$$M = \sum_{k=0}^{\infty} a_k L^k$$

we associate to M the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k.$$

We call $f(t)$ the *L-indicator* of the linear functional M. When L is the generator A, we call $f(t)$ simply the *indicator* of M.

Recall that the algebra $\mathbf{F}$ of formal power series can be made into a topological algebra by stipulating that a sequence $f_n(t)$ converges whenever the sequence of coefficients of each power of t converges in the discrete topology of the field K; that is, whenever the sequence of coefficients is eventually constant. In this topology we can show

THEOREM 3. *Let L be a delta functional. Then the mapping ϕ which associates to every linear functional*

$$M = \sum_{k=0}^{\infty} a_k L^k, \qquad a_k \in K,$$

the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

is a continuous isomorphism of the umbral algebra onto the algebra of formal power series.

Proof. The Expansion Theorem, together with Corollary 1, shows that ϕ is linear, one-to-one and onto. Corollary 2 shows that ϕ is an algebra homomorphism.

To prove that ϕ is continuous, suppose L has associated sequence $p_n(x)$, and suppose M_n is a sequence of linear functionals converging to the linear functional M. If

$$M_n = \sum_{k=0}^{\infty} \alpha_k^{(n)} L^k$$

and

$$M = \sum_{k=0}^{\infty} \alpha_k L^k$$

we must show that

$$\phi(M_n) = \sum_{k=0}^{\infty} \alpha_k^{(n)} t^k$$

converges to

$$\phi(M) = \sum_{k=0}^{\infty} \alpha_k t^k.$$

By definition of convergence in P^*, for any fixed $j \geqslant 0$, there is an n_0 such that $n > n_0$ implies $\langle M_n \mid p_j(x)\rangle = \langle M \mid p_j(x)\rangle$. In other words, $n > n_0$ implies $\alpha_j^{(n)} = \alpha_j$. But this is the definition of convergence in $\mathbf{F}$, and thus $\phi(M_n)$ converges to $\phi(M)$.

COROLLARY 1. *A linear functional M is a delta functional if and only if the L-indicator of M has zero constant term and nonzero linear term.*

COROLLARY 2. *A linear functional M is a delta functional if and only if, for every delta functional L, there exists an invertible functional N such that $M = LN$.*

The following property of delta functionals will be repeatedly used:

PROPOSITION 4.2. *Let L be a linear functional with $\langle L \mid 1\rangle = 0$. Then the powers of L, including $L^0 = \epsilon$, span the space P^* if and only if L is a delta functional.*

Proof. If L is a delta functional, the Expansion Theorem shows that the powers of L span P^*. Conversely, suppose the powers of L span P^*. If $\langle L \mid x\rangle = 0$, then $\langle L^k \mid x\rangle = 0$ for all $k \geqslant 0$. But since $\langle A \mid x\rangle \neq 0$, the generator A cannot lie in the span of L^k. Thus $\langle L \mid x\rangle \neq 0$.

We now turn to the final main result of this section, which is another one-to-one correspondence between delta functionals and sequences of binomial type. We begin with a characterization of the coefficients of sequences of binomial type. Its verification is straightforward.

PROPOSITION 4.3. *A polynomial sequence*

$$q_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

is of binomial type if and only if

$$\binom{i+j}{i} a_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} a_{k,i} a_{n-k,j} \qquad (***)$$

for all $n \geqslant 0$, *and for all* $i, j \geqslant 0$.

We define the *conjugate sequence* of a delta functional L as the polynomial sequence

$$q_n(x) = \sum_{k \geqslant 0} \frac{\langle L^k \mid x^n \rangle}{k!} x^k.$$

By Proposition 3.5, each $q_n(x)$ is a polynomial of degree n.

THEOREM 4. (a) *Every conjugate sequence is a sequence of binomial type.*

(b) *Every sequence of binomial type is a conjugate sequence.*

Proof. (a) It follows directly from the definition of product of linear functionals that the coefficients of the conjugate sequence satisfy (***), thus proving part (a).

(b) Given a sequence $q_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ of binomial type, we define a sequence of linear functionals L_k by

$$\langle L_k \mid x^n \rangle = k!\, a_{n,k}\,.$$

Then $\langle L_1 \mid 1 \rangle = a_{1,0} = 0$ and $\langle L_1 \mid x \rangle = a_{1,1} \neq 0$ so L_1 is a delta functional. Moreover, since the $a_{n,k}$ satisfy (***), we infer that

$$\langle L_{i+j} \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_i \mid x^k \rangle \langle L_j \mid x^{n-k} \rangle.$$

Therefore $L_{i+j} = L_i L_j$. This implies that $L_k = L_1{}^k$ and $q_n(x)$ is the conjugate sequence for L_1, proving part (b).

Thus we see that a delta functional L is associated with two sequences of binomial type, its associated sequence $p_n(x)$ and its conjugate sequence $q_n(x)$. We will say that $q_n(x)$ is *reciprocal* to $p_n(x)$. Should $p_n(x) = q_n(x)$, as in the case $L = A$, the sequence $p_n(x)$ is called *self-reciprocal.*

Similarly, a sequence $p_n(x)$ of binomial type is associated with two delta functionals, namely, the functional L, for which $p_n(x)$ is the associated sequence and the functional $\tilde{L}$, for which $p_n(x)$ is the conjugate sequence. We will say that $\tilde{L}$ is *reciprocal* to L. Should $L = \tilde{L}$, the linear functional L is called *self-reciprocal.*

If $p_n(x)$ is a sequence of binomial type, and if L is the linear functional satisfying

$$\langle L \mid p_n(x)\rangle = \delta_{n,1}$$

for $n \geqslant 0$, then by the spanning argument, L is the delta functional whose associated sequence is $p_n(x)$. Thus

$$\langle L^k \mid p_n(x)\rangle = n!\, \delta_{n,k}\,.$$

We generalize this with:

PROPOSITION 4.4. *Let $p_n(x)$ be a sequence of binomial type. Let L be a delta functional and let M be an invertible linear functional. Then $p_n(x)$ is the associated sequence for LM^{-1} if and only if*

$$\langle L \mid p_n(x)\rangle = n\langle M \mid p_{n-1}(x)\rangle$$

for $n \geqslant 1$.

Proof. If $p_n(x)$ is the associated sequence for LM^{-1}, we have

$$\begin{aligned}\langle L \mid p_n(x)\rangle &= \langle (LM^{-1})\, M \mid p_n(x)\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle LM^{-1} \mid p_k(x)\rangle\langle M \mid p_{n-k}(x)\rangle \\ &= n\langle M \mid p_{n-1}(x)\rangle,\end{aligned}$$

the last equality since $\langle LM^{-1} \mid p_k(x)\rangle = \delta_{k,1}$. Conversely, if $\langle L \mid p_n(x)\rangle = n\langle M \mid p_{n-1}(x)\rangle$, for $n \geqslant 1$, we have

$$\begin{aligned}\langle LM^{-1} \mid p_n(x)\rangle &= \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle\langle M^{-1} \mid p_{n-k}(x)\rangle \\ &= \sum_{k=1}^{n} \binom{n}{k} k\langle M \mid p_{k-1}(x)\rangle\langle M^{-1} \mid p_{n-k}(x)\rangle \\ &= n\langle MM^{-1} \mid p_{n-1}(x)\rangle = n\langle \epsilon \mid p_{n-1}(x)\rangle \\ &= n\delta_{n,1} = \delta_{n,1}\,.\end{aligned}$$

By the remark preceding the proposition, $p_n(x)$ is the associated sequence for LM^{-1}.

5. Examples

We begin a continuing discussion of some notable examples. We label each installment by the symbol *a.b*, where *a* is the example number and *b* is the installment number.

First we give examples of delta functionals, along with their associated sequences, indicators, and some applications of the Expansion Theorem. Derivation of the associated sequences is deferred to Section 8, and computation of the indicators, being straightforward, is omitted.

1.1. The sequence x^n is the associated sequence for the generator A, whose indicator is the formal power series t. The binomial identity is the binomial formula, and expansion of the evaluation ϵ_y in powers of the generator is Taylor's formula

$$\epsilon_y = \sum_{k\geq 0} \frac{y^k}{k!} A^k$$

since $\langle A^k \mid p(x)\rangle = p^{(k)}(0)$.

2.1. The *falling factorial sequence* $(x/a)_n$, where $(y)_n = y(y-1)\cdots(y-n+1)$ is the falling factorial, is the associated sequence for the *forward difference functional* $\epsilon_a - \epsilon$, whose indicator is the formal power series $e^{at}-1$.
For $a = 1$, the binomial identity becomes

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}\,.$$

Expansion of the evaluation ϵ_y in terms of the forward difference functional gives Newton's expansion

$$\epsilon_y = \sum_{k\geq 0} \frac{(y/a)_k}{k!} (\epsilon_a - \epsilon)^k. \qquad (*)$$

Using the expansion

$$(\epsilon_a - \epsilon)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \epsilon_a{}^j$$

and the fact that $\epsilon_a{}^j = \epsilon_{aj}$, and applying the result to a polynomial $p(x)$ gives

$$p(y) = \sum_{k\geq 0} \binom{y/a}{k} \sum_{j=0}^{n} \binom{k}{j} (-1)^{k-j} p(aj).$$

By way of orientation, we derive one of the classical formulas for numerical differentiation. This results from the expansion of A in powers of $\epsilon_a - \epsilon$:

$$p'(0) = \langle A \mid p(x) \rangle = \sum_{k \geq 0} \frac{\langle A \mid (x/a)_k \rangle}{k!} \langle (\epsilon_a - \epsilon)^k \mid p(x) \rangle$$

$$= \sum_{k \geq 0} \frac{(-1)^{k+1}}{ak} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} p(aj).$$

3.1. The *rising factorial sequence* $\langle x/a \rangle_n$, where $\langle y \rangle_n = y(y+1) \cdots (y+n-1)$, is the associated sequence for the *backward difference functional* $\epsilon - \epsilon_{-a}$, whose indicator is the formal power series $1 - e^{-at}$. The identities are similar to those of the forward difference functional, and we mention only

$$\langle x + y \rangle_n = \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_k \langle y \rangle_{n-k}.$$

If L is any delta functional, the *Abelization* of L is the delta functional $\epsilon_a L$. The associated polynomials for $\epsilon_a L$ can be explicitly computed (Section 7) in terms of the associated polynomials for L. We give two examples.

4.1. The Abel polynomials $A_n(x, a) = x(x - an)^{n-1}$ are easily verified to be the associated polynomials for the *Abel functional* $\epsilon_a A$, where

$$\langle \epsilon_a A \mid p(x) \rangle = p'(a).$$

The indicator of the Abel functional is the series te^{at}.

Theorem 2 gives a proof of Abel's identity:

$$(x + y)(x + y - an)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} xy(x - ak)^{k-1} (y - a(n-k))^{n-k-1}.$$

Expansion of the evaluation ϵ_y in powers of $\epsilon_a A$ gives

$$\epsilon_y = \sum_{k \geq 0} \frac{y(y - ak)^{k-1}}{k!} A^k \epsilon_{ka},$$

or

$$p(y) = \sum_{k \geq 0} \frac{y(y - ak)^{k-1}}{k!} p^{(k)}(ka),$$

and, when $p(y) = e^y$, we obtain the beautiful

$$e^y = \sum_{k \geq 0} \frac{y(y - ak)^{k-1}}{k!} e^{ka},$$

which is easily justified by a limiting process.

5.1. The *Gould polynomials*

$$G_n(x, a, b) = \frac{x}{x - an}\binom{\frac{x - an}{b}}{n}$$

are the associated polynomials for the delta functional $\epsilon_a(\epsilon_b - \epsilon)$, the *difference-Abel functional*, whose indicator is $e^{at}(e^{bt} - 1)$.

The binomial identity, resulting from Theorem 2, is Vandermonde convolution

$$\frac{x+y}{x+y-an}\binom{(x+y-an)/b}{n}$$
$$= \sum_{k=0}^{n} \frac{x}{x-ak}\,\frac{y}{y-a(n-k)}\binom{(x-ak)/b}{k}\binom{(y-a(n-k))/b}{n-k}.$$

Corollary 5 to Theorem 1 gives the interesting expansion

$$p(y) = \sum_{k\geqslant 0} \frac{y}{y-ak}\binom{(y-ak)/b}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}p(ak+bj).$$

6.1. The *central difference functional* $\delta_a = \epsilon_{a/2} - \epsilon_{-a/2}$, whose indicator is the series $e^{at/2} - e^{-at/2} = 2\sinh at/2$, is a special case of the preceding example. For $a = 1$, the associated sequence

$$x^{[n]} = x(x + n/2 - 1)_{n-1}$$

gives the *Steffensen polynomials.*

By expanding a polynomial $p(x)$ in terms of the Steffensen polynomials (Corollary 5 to Theorem 1), we obtain the interpolation formula

$$p(y) = \sum_{k\geqslant 0} \frac{y}{y+k/2}\binom{y+k/2}{k}\langle \delta^k \mid p(x)\rangle.$$

7.1. The (basic) Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-x)^k$$

are the associated polynomials for the *Laguerre functional*

$$\langle l \mid p(x)\rangle = \int_{-\infty}^{0} e^t p'(t)\,dt.$$

From $\langle l \mid x^n\rangle = -n!$, we infer that the indicator of l is the formal power series $t/(t-1)$.

Expanding the polynomial $p_n(x) = x^n$ in terms of the Laguerre polynomials gives the remarkable

$$y^n = \sum_{k\geqslant 0} (-1)^k \frac{n!}{k!} \binom{n-1}{k-1} L_k(y).$$

We now use Proposition 3.4 to derive some identities. Taking, for example, all $L_i = \epsilon_a - \epsilon$, the forward difference functional, and $p_n(x)$ any sequence of binomial type, we find

$$\langle(\epsilon_a - \epsilon)^k \mid p_n(x)\rangle = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} p_{i_1}(a) \cdots p_{i_k}(a).$$

Expanding $(\epsilon_a - \epsilon)^k$ by the binomial theorem

$$(\epsilon_a - \epsilon)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \epsilon_{ia}$$

gives the identity

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} p_n(ia) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} p_{i_1}(a) \cdots p_{i_k}(a).$$

For $p_n(x) = x^n$, this specializes to

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k}.$$

The right side counts the number of ways of placing n balls into k boxes, with no box empty. It thus equals $k!\, S(n, k)$, where $S(n, k)$ are the Stirling numbers of the second kind.

Setting $p_n(x) = (x/b)_n$, the falling factorial sequence, and then replacing a/b by r, we obtain the binomial identity:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \binom{ir}{n} = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{r}{i_1} \cdots \binom{r}{i_k}$$

valid for all $r > 0$. With $p_n(x) = \langle x/b\rangle_n$, the rising factorial sequence, a similar identity is obtained where the multiset coefficients replace the binomial coefficients.

The difference–Abel functional $\epsilon_a(\epsilon_b - \epsilon)$ gives other remarkable identities by the same use of Proposition 3.4. For example,

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} p_n(ak + bi)$$

$$= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1, \ldots, i_k} [p_{i_1}(a + b) - p_{i_1}(a)] \cdots [p_{i_k}(a + b) - p_{i_k}(a)]$$

for any sequence $p_n(x)$ of binomial type. In particular, for $p_n(x) = x^n$ with $a + b = -1$ and $a = 1$:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (k - 2i)^n$$

$$= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1, \ldots, i_k} [(-1)^{i_1} - 1] \cdots [(-1)^{i_k} - 1]$$

$$= (-2)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \text{odd}}} \binom{n}{i_1, \ldots, i_k}.$$

Except for the factor $(-2)^k$, the right side counts the number of ways of placing n balls into k boxes, subject to the condition that each box contain an odd number of balls.

We next consider some examples of conjugate sequences.

1.2. The conjugate sequence for the generator A is clearly the sequence x^n.

2.2. The conjugate sequence for the forward difference functional $\epsilon_a - \epsilon$ can be obtained from $\langle(\epsilon_a - \epsilon)^k \mid x^n\rangle = a^k k!\, S(n, k)$, where $S(n, k)$ are the Stirling numbers of the second kind. In fact, for $a = 1$:

$$\phi_n(x) = \sum_{k=0}^{n} S(n, k)\, x^k.$$

These are the exponential polynomials.

3.2. The backward difference functional gives a variant of the exponential polynomials, namely, for $a = 1$:

$$q_n(x) = \sum_{k=0}^{n} (-1)^{n-k} S(n, k)\, x^k.$$

4.2. The conjugate sequence for the Abel functional $\epsilon_a A$ is easily computed to be

$$\mu_n(x, a) = \sum_{k=0}^{n} \binom{n}{k} (ak)^{n-k}\, x^k.$$

5.2. The conjugate sequence for the difference-Abel functional $\epsilon_a(\epsilon_b - \epsilon)$ has not occurred in the literature. It is:

$$g_n(x, a, b) = \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} (ak)^i\, b^{n-i} S(n-i, k)\, x^k.$$

We call these the *conjugate Gould polynomials.*

6.2. The conjugate polynomials for the central difference functional δ_a are found by the same methods to be the *Carlitz–Riordan polynomials*

$$K_n(x) = a^n \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-k-i}\, k^i 2^{n-i} S(n-i, k)\, x^k.$$

7.2. For the Laguerre functional l, we find

$$\langle l^k \mid x^n \rangle = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} (-1)^k\, i_1! \cdots i_k!$$
$$= (-1)^k\, n! \binom{n-1}{k-1}.$$

Thus the conjugate sequence of the Laguerre functional is the same as the associated sequence, namely, the basic Laguerre polynomials. The explanation of this remarkable fact is given in the sequel.

8.2. *The Bell polynomials.* For the first time we require a field other than the real or complex field. Let k be a field of characteristic zero, to which a sequence of independent transcendentals $x_1, x_2, \ldots$ has been adjoined. Over this field, define the *generic delta functional* L by

$$\langle L \mid x^n \rangle = x_n \qquad \text{for} \quad n \geqslant 1,$$
$$\langle L \mid 1 \rangle = 0.$$

The conjugate polynomials for the generic delta functional are the Bell polynomials. An explicit formula for the coefficients is obtained from Proposition 3.4:

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots) = \frac{\langle L^k \mid x^n \rangle}{k!} = \sum \frac{n!}{c_1!\, c_2! \cdots} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots,$$

where the sum ranges over all nonnegative integers $c_1, c_2, \ldots$ satisfying $c_1 + 2c_2 + \cdots = n$ and $c_1 + c_2 + \cdots = k$.

For the Bell polynomials, we use the notation

$$b_n(x; x_1, x_2, \ldots) = \sum_{k=0}^{n} B_{n,k} x^k.$$

All known identities for the Bell coefficients $B_{n,k}$ follow from the multiplication rules for delta functionals. We give a sampling:

(a) $$kB_{n,k} = \sum_{j=k-1}^{n-1} \binom{n}{j} x_{n-j} B_{j,k-1},$$

rewritten in the present notation, becomes the trivial

$$\frac{\langle L^k \mid x^n \rangle}{(k-1)!} = \sum_{j=k-1}^{n-1} \binom{n}{j} \langle L \mid x^{n-j} \rangle \frac{\langle L^{k-1} \mid x^j \rangle}{(k-1)!}.$$

(b) Let the delta functional L_1 be defined by $\langle L_1 \mid x^n \rangle = x_{n+1}/(n+1)$, for $n \geqslant 1$ and $\langle L_1 \mid 1 \rangle = 0$. Then $L = L_1 A + x_1 A$. The conjugate sequence for L_1 is $b_n(x; x_2/2, x_3/3, \ldots)$. An identity relating this polynomial sequence to the Bell polynomials is derived as follows. We apply

$$L^k = \sum_{j=0}^{k} \binom{k}{j} x_1^{k-j} L_1^{\,j} A^k$$

to the polynomial x^n and simplify:

$$\langle L^k \mid x^n \rangle = \sum_{j=0}^{k} \binom{k}{j} x_1^{k-j} (n)_k \langle L_1^{\,j} \mid x^{n-k} \rangle.$$

Hence

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{j=0}^{k} \frac{n!}{(n-k)!\,(k-j)!} x_1^{k-j} B_{n-k,j}(x_2/2, x_3/3, \ldots).$$

Similar identities can be obtained with the unique delta functional L_i such that $L = x_1 A + x_2 A^2/2! + \cdots + x_{i-1} A^{i-1}/(i-1)! + A^i L$.

(c) Consider now the field k with additional independent transcendentals $y_1, y_2, \ldots$ adjointed. The conjugate sequence of the delta functional L' given by $\langle L' \mid x^n \rangle = x_n + y_n$ is $b_n(x; x_1 + y_1, x_2 + y_2, \ldots)$. Setting $\langle L'' \mid x^n \rangle = y_n$, so that $L' = L + L''$, one obtains

$$\langle (L')^k \mid x^n \rangle = \sum_{j=0}^{k} \binom{k}{j} \langle L^j \mid x^n \rangle \langle (L'')^{k-j} \mid x^n \rangle,$$

whence we obtain

$$B_{n,k}(x_1 + y_1, x_2 + y_2, \ldots) = \sum_{j=0}^{k} B_{n,j}(x_1, x_2, \ldots)\, B_{n,k-j}(y_1, y_2, \ldots).$$

(d) From Proposition 3.4, one easily obtains

$$B_{n,k}(0, 0, \ldots, x_j, 0, \ldots) = 0,$$

unless $n = jk$, and

$$B_{jk,k} = \frac{(jk)!}{k!\,(j!)^k} x_j .$$

(e) Every delta functional can be obtained from the Bell generic delta functional by specializing the values of the x_i. Thus every formula for the Bell polynomials gives a formula for all conjugate sequence. For example, from (b) one obtains

$$\langle L^k \mid x^n\rangle = \sum_{j=0}^{k} \binom{k}{j} (n)_k \langle L \mid x\rangle^{k-j} \langle L_1{}^j \mid x^{n-k}\rangle,$$

where L is any delta functional and where $L = L_1 A + x_1 A$. Similarly, (c) gives the conjugate polynomials of the sum of two (or more) delta functionals in terms of the conjugate sequences of the summand.

6. Automorphisms and Derivations

Given two polynomial sequences $p_n(x)$ and $q_n(x)$, a frequently encountered problem is that of determining a matrix of constants $c_{n,k}$, which we call the *connection constants* of $p_n(x)$ with $q_n(x)$, such that

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} p_k(x). \qquad (*)$$

In this section, we give a solution to this problem when the polynomial sequences are of binomial type. The solution we propose takes a particularly simple form in the *umbral notation* we now introduce. If $r(x) = \sum_{k=0}^{n} c_k x^k$ is a polynomial, and $p_n(x)$ is a polynomial sequence, the *umbral composition* of $r(x)$ with $p_n(x)$ is the polynomial, written $r(\mathbf{p}(x))$, and defined by

$$r(\mathbf{p}(x)) = \sum_{k=0}^{n} c_k p_k(x).$$

If $r_n(x)$ and $p_n(x)$ are two polynomial sequences, the umbral composition of $r_n(x)$ with $p_n(x)$ is the polynomial sequence $r_n(\mathbf{p}(x))$. In this notation, $(*)$ becomes

$$q_n(x) = r_n(\mathbf{p}(x)),$$

where $r_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$.

Umbral composition is simply the result of applying a suitable linear operator to a polynomial sequence. In particular, if α is the linear operator on P defined by $\alpha x^n = p_n(x)$ for $n = 0, 1, 2,\ldots$, then $\alpha r_n(x) = r_n(\mathbf{p}(x))$, and $(*)$ becomes

$$q_n(x) = \alpha r_n(x).$$