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# The Theory of the Umbral Calculus II*

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## 1. INTRODUCTION

This is the second in a series of papers intended to develop the theory of the umbral calculus. We shall assume familiarity with Sections 1–5 of the first paper, hereinafter denoted by UCI.

In UCI we studied a certain class of polynomial sequences, called Sheffer sequences, by studying the dual vector space $P^*$ to the algebra of polynomials $P$. First we made $P^*$ into an algebra—the algebra of formal-power series in $t$. Sheffer sequences were defined as those sequences in $P$ which are orthogonal to some geometric sequence $g(t)f(t)^k$ in $P^*$

$$\langle g(t)f(t)^k | s_n(x)\rangle = c_n \delta_{n,k}.$$

The degree requirements $\deg g(t) = 0$ and $\deg f(t) = 1$ were placed in order to ensure the existence and uniqueness of the sequence $s_n(x)$.

Our present goal is to extend the theory to a more general class of sequences in $P^*$ than the class of geometric sequences. This will bring some important new polynomial sequences into the purview of the umbral calculus. We shall call elements of this new class decentralized geometric sequences.

Let us illustrate with a simple example. Recall that if $c_n = n!$ for all $n \geqslant 0$, then the evaluation functional $\varepsilon_y(t)$ is the exponential series $e^{yt}$,

$$\langle e^{yt} | p(x)\rangle = p(y).$$

Now if $y_0, y_1,\ldots$ is a sequence of independent transcendentals (variables), then one decentralization of the geometric sequence $t^k$ is the sequence $e^{y_k t} t^k$. The intuitive reason for this is that

$$\langle t^k | p(x)\rangle = p^{(k)}(0)$$

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and

$$\langle e^{y_k t} t^k \mid p(x) \rangle = p^{(k)}(y_k).$$

In words, one decentralization of the sequence "$k$th derivative evaluated at 0" is the sequence "$k$th derivative evaluated at $y_k$." Now the sequence $x^n$ in $P$ is orthogonal to the sequence $t^k$,

$$\langle t^k \mid x^n \rangle = n! \delta_{n,k}.$$

Thus, one decentralization of the sequence $x^n$ is the sequence $G_n(x)$ satisfying

$$\langle e^{y_k t} t^k \mid G_n(x) \rangle = n! \delta_{n,k}$$

or rather,

$$G_n^{(k)}(y_k) = n! \delta_{n,k}.$$

The polynomials $G_n(x)$ are known as the Gončarov (or Gontscharoff) polynomials and are of considerable importance in the theory of interpolation. Thus, the umbral calculus may be extended to include the Gončarov polynomials. Another important polynomial sequence which we shall incorporate in this way is the sequence

$$s_n(x) = (x - y_0)(x - y_1) \cdots (x - y_{n-1}).$$

Up to now we have been using the algebra of formal power series in a single variable to study the dual vector space $P^*$ and hence also the algebra $P$. We have come to a point in the development of the umbral calculus where it pays to reverse this point of view. Accordingly, we devote a section of this paper to the study of the algebra of formal power series in a single variable by thinking of formal power series as linear functionals on $P$.

## 2. Decentralized Geometric Sequences

To fix our terminology we shall let $P$ be the algebra of polynomials in $x$ and $\mathscr{F}$ be the algebra of formal power series in $t$, both over the base field $K$ of characteristic zero. We let $c_n$ be a sequence of nonzero constants. The algebra $\mathscr{F}$ represents the set of all linear functionals on $P$ as well as certain set of linear operators on $P$—this by means of

$$\langle t^k \mid x^n \rangle = c_n \delta_{n,k}$$

and

$$t^k x^n = (c_n/c_{n-k})\, x^{n-k}.$$

The details may be found in UCI.

We would like to remind the reader of an extremely useful result.

LEMMA 2.1. *If* $\deg f_k(t) = k$ *and* $\langle f_k(t) | p(x) \rangle = 0$ for all $k \geqslant 0$, then $p(x) = 0$. Similarly, if $\deg p_k(x) = k$ *and* $\langle f(t) | p_k(x) > = 0$ *for all* $k \geqslant 0$, *then* $f(t) = 0$.

Suppose

$$y_0, y_1, y_2, \ldots \qquad \text{and} \qquad z_0, z_1, z_2, \ldots$$

form a set of independent transcendentals (i.e., variables). Let $K$ be the quotient field of the algebra of all formal power series in these transcendentals over a base field $C$. We need not concern ourselves with a precise description of $K$. Our only concern is that $K$ is a field. It is critical to the theory to assume that the constants $c_n$ lie in $C$ and, hence, are independent of the transcendentals. For any integer $k$ we define the function $S^k$ on $K$ which shifts the subscripts of the transcendentals. Thus,

$$S^k y_j = y_{j+k}, \qquad S^k z_j = z_{j+k}.$$

Notice that $S^k y_j$ is not defined if $k + j < 0$ and we shall be careful to avoid such a contigency. Now suppose $f(t)$ is a series in $\mathscr{F}$. Then for $k > 0$ we shall write $S^k f(t)$ as $f(t; k)$. For uniformity we may write $f(t)$ as $f(t; 0)$. To put it in words, $f(t)$ may involve some transcendentals in its coefficients. Then $f(t; k)$ is obtained from $f(t)$ by adding $k$ to the subscripts of these transcendentals. We adopt a similar notation $S^k p(x) = p(x; k)$ for polynomials in $P$.

A *decentralized geometric sequence* in $\mathscr{F}$ is a sequence of the form

$$h_k(t) = g(t; k) f(t; 0) \cdots f(t; k-1), \qquad h_0(t) = g(t; 0),$$

where $g(t)$ and $f(t)$ are in $\mathscr{F}$. Whenever $g(t)$ and $f(t)$ do not involve the transcententals the sequence $h_k(t)$ reduces to a geometric sequence in $\mathscr{F}$.

We shall have frequent occasion to use the fact that

$$S^k \langle f(t) | p(x) \rangle = \langle S^k f(t) | S^k p(x) \rangle \tag{2.1}$$

and

$$S^k [f(t)\, p(x)] = [S^k f(t)][S^k p(x)] \tag{2.2}$$

whenever all expressions are defined. In particular, if $\langle f(t) | p(x) \rangle$ is independent of the transcendentals, then $\langle f(t) | p(x) \rangle = \langle f(t; k) | p(x; k) \rangle$ for all $k \geqslant 0$.

## 3. Decentralized Sheffer Sequences

As usual by a *sequence* $p_n(x)$ in $P$ we imply that $\deg p_n(x) = n$. Recall that a delta series $f(t)$ in $\mathscr{F}$ is a series satisfying $\deg f(t) = 1$ and an invertible series $g(t)$ is a series satisfying $\deg g(t) = 0$.

THEOREM 3.1. *Let $f(t)$ be a delta series and let $g(t)$ be invertible. Then the identity*

$$\langle g(t; k) f(t; 0) \cdots f(t; k-1) | s_n(x) \rangle = c_n \delta_{n,k} \tag{3.1}$$

*for all $n, k \geqslant 0$ determines a unique sequence $s_n(x)$ in $P$.*

*Proof.* Since $\deg g(t; k) f(t; 0) \cdots f(t; k-1) = k$ the sequence $g(t; k) f(t; 0) \cdots f(t; k-1)$ forms a pseudobasis for $\mathscr{F}$ and the proof is virtually identical to that given for Theorem 5.1 of UCI.

We will call the sequence $s_n(x)$ the *decentralized Sheffer sequence* for the pair $(g(t), f(t))$, or more briefly, we say $s_n(x)$ is *decentralized Sheffer* for $(g(t), f(t))$.

THEOREM 3.2 (The expansion theorem). *Let $s_n(x)$ be decentralized Sheffer for $(g(t), f(t))$. Then for any $h(t)$ in $\mathscr{F}$,*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{c_k} g(t; k) f(t; 0) \cdots f(t; k-1).$$

*Proof.* We simple apply the right side to $s_n(x)$ to obtain $\langle h(t) | s_n(x) \rangle$. An application of Lemma 2.1 completes the proof.

COROLLARY 3. *Let $s_n(x)$ be decentralized Sheffer for $(g(t), f(t))$. Then for any $p(x)$ in $P$,*

$$p(x) = \sum_{k \geqslant 0} \frac{\langle g(t; k) f(t; 0) \cdots f(t; k-1) | p(x) \rangle}{c_k} s_k(x).$$

We now give an operator characterization of decentralized Sheffer sequence.

THEOREM 3.3. *The sequence $s_n(x)$ is decentralized Sheffer for $(g(t), f(t))$ if and only if*

(1) $\langle g(t) | s_n(x) \rangle = c_n \delta_{n,0}$

(2) $f(t)\, s_n(x) = (c_n/c_{n-1})\, s_{n-1}(x; 1)$.

*Proof.* Suppose $s_n(x)$ is decentralized Sheffer for $(g(t), f(t))$. Then Eq. (3.1) for $k=0$ gives (1). To prove (2) we have for $k \geqslant 1$

$$\begin{aligned}
&\langle g(t; k) f(t; 1) \cdots f(t; k-1) | f(t; 0)\, s_n(x) \rangle \\
&\quad = c_n \delta_{n,k} \\
&\quad = (c_n/c_{n-1})\, c_{n-1} \delta_{n-1,k-1} \\
&\quad = (c_n/c_{n-1}) \langle g(t; k-1) f(t; 0) \cdots f(t; k-1) | s_{n-1}(x) \rangle \\
&\quad = (c_n/c_{n-1}) \langle g(t; k) f(t; 1) \cdots f(t; k-1) | s_{n-1}(x; 1) \rangle.
\end{aligned}$$

Thus, by Lemma 2.1 we must have $f(t; 0)\, s_n(x) = (c_n/c_{n-1})\, s_{n-1}(x; 1)$. For the converse suppose (1) and (2) hold. Then

$$\begin{aligned}
&\langle g(t; k) f(t; 0) \cdots f(t; k-1) | s_n(x) \rangle \\
&\quad = (c_n/c_{n-1}) \langle g(t; k) f(t; 1) \cdots f(t; k-1) | s_{n-1}(x; 1) \rangle \\
&\quad = (c_n/c_{n-1})\, S^1 \langle g(t; k-1) f(t; 0) \cdots f(t; k-2) | s_{n-1}(x) \rangle
\end{aligned}$$

and continuing in the way we obtain

$$\begin{aligned}
&(c_n/c_{n-k})\, S^k \langle g(t; 0) | s_{n-k}(x) \rangle \\
&\quad = (c_n/c_{n-k})\, S^k c_{n-k} \delta_{n,k} \\
&\quad = c_n \delta_{n,k}.
\end{aligned}$$

This completes the proof.

We now come to the decentralized Sheffer identity.

THEOREM 3.4. *A sequence $s_n(x)$ is decentralized Sheffer for $(g(t), f(t))$ for some delta series $f(t)$ if and only if* $\deg g(t) = 0$ *and*

$$\varepsilon_y(t)\, s_n(x) = \sum_{k=0}^{n} \frac{c_n}{c_k c_{n-k}}\, s_k(y)\, g(t; k)\, s_{n-k}(x; k) \tag{3.2}$$

*for all $y$ in $K$ and for all $n \geqslant 0$.*

*Proof.* Suppose $s_n(x)$ is decentralized Sheffer for $(g(t), f(t))$. The expansion theorem for $h(t) = \varepsilon_y(t)$ gives

$$\varepsilon_y(t) = \sum_{k=0}^{\infty} \frac{s_k(y)}{c_k} g(t; k) f(t; 0) \cdots f(t; k-1).$$

Applying both sides of this to $s_n(x)$ and using Theorem 3.3 and Eq. (2.2) gives the Sheffer identity. For the converse we define the linear operator $T$ on $P$ by

$$Ts_n(x) = (c_n/c_{n-1})\, s_{n-1}(x; 1), \qquad Ts_0(x) = 0. \tag{3.3}$$

Now since the left side of (3.2) is easily seen to be symmetric in $x$ and $y$ we have

$$\varepsilon_y(t)\, s_n(x) = \sum_{k=0}^{\infty} \frac{c_n}{c_k c_{n-k}} s_k(x)\, g_y(t; k)\, s_{n-k}(y; k), \tag{3.4}$$

where the subscript $y$ indicates that the operator $g_y(t; k)$ acts on $y$. From (3.4) we obtain, for $y$ in $C$,

$$\begin{aligned}
T\varepsilon_y(t)\, s_n(x) &= T \sum_{k=0}^{n} \frac{c_n}{c_k c_{n-k}} s_k(x)\, g_y(t; k)\, s_{n-k}(y; k) \\
&= \sum_{k=1}^{n} \frac{c_n}{c_{k-1} c_{n-k}} s_{k-1}(x; 1)\, g_y(t; k)\, s_{n-k}(y; k) \\
&= \sum_{k=0}^{n-1} \frac{c_n}{c_k c_{n-k-1}} s_k(x; 1)\, g_y(t; k+1)\, s_{n-k-1}(y; k) \\
&= S^1 \sum_{k=0}^{n-1} \frac{c_n}{c_k c_{n-k-1}} s_k(x)\, g_y(t; k)\, s_{n-k-1}(y; k) \\
&= (c_n/c_{n-1})\, S^1 \varepsilon_y(t)\, s_{n-1}(x) \\
&= (c_n/c_{n-1})\, \varepsilon_y(t)\, s_{n-1}(x; 1) \\
&= \varepsilon_y(t)\, Ts_n(x).
\end{aligned}$$

Thus, $T\varepsilon_y(t) = \varepsilon_y(t)\, T$ for $y$ in $C$. By Corollary 2 to Proposition 4.1 in UCI we deduce the existence of a series $f(t)$ in $\mathscr{F}$ for which $T = f(t)$. Since $s_n(x)$ is assumed to be a sequence in $P$ the definition of $T$ implies that $f(t)$ is a delta series. Equations (3.3) and (3.4) imply

$$\varepsilon_y(t)\, s_n(x) = \sum_{k=0}^{n} \frac{s_k(y)}{c_k} g(t; k) f(t; 0) \cdots f(t; k-1)\, s_n(x).$$

Since this holds for all $n \geqslant 0$ we have

$$\varepsilon_y(t) = \sum_{k=0}^{n} \frac{s_k(y)}{c_k} g(t;k) f(t;0) \cdots f(t;k-1).$$

Applying these linear functionals to $s_n(x)$ gives

$$s_n(y) = \sum_{k=0}^{n} \frac{\langle g(t;k) f(t;0) \cdots f(t;k-1) | s_n(x) \rangle}{c_k} s_k(y)$$

and comparing coefficients of $s_k(x)$ gives

$$\langle g(t;k) f(t;0) \cdots f(t;k-1) | s_n(x) \rangle = c_n \delta_{n,k}$$

which concludes the proof.

The reader may recall that a polynomial sequence $s_n(x)$ satisfying the binomial identity

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, s_{n-k}(y)$$

has been termed a sequence of binomial type. Setting $c_n = n!$ and $g(t;k) = e^{y_k t}$ we find that (3.2) becomes

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y)\, s_{n-k}(x + y_k; k).$$

One sees the justification in calling this the *decentralized binomial identity.*

## 4. Computation of Decentralized Sheffer Sequence

In order to derive formulas for the computation of decentralized Sheffer sequences we are required to make some slight extensions in the theory. Let $f(t)$ be a delta series. Then $f(t)$ has no multiplicative inverse in $\mathscr{F}$. Thought of as a formal Laurent series, however, $f(t)$ has a multiplicative inverse, denoted by $f^{-1}(t)$ and satisfying

$$f^{-1}(t) = t^{-1}(f(t)/t)^{-1}.$$

We shall require the use of the linear operator $f^{-1}(t)$ on $P$ defined by extending the definition

$$t^k x^n = (c_n/c_{n-k})\, x^{n-k}$$

for all $k \geqslant 0$ to include

$$t^{-1}x^n = (c_n/c_{n+1})\, x^{n+1}.$$

To be absolutely clear, if $f^{-1}(t) = \sum_{k=-1}^{\infty} a_k t^k$, then

$$\begin{aligned} f^{-1}(t)\, x^n &= \sum_{k=-1}^{\infty} a_k t^k x^n \\ &= \sum_{k=-1}^{\infty} a_k \frac{c_n}{c_{n-k}} x^{n-k}. \end{aligned}$$

We shall also require the use of the linear functional $f^{-1}(t)$. In this case we set

$$\langle t^{-1} | x^n \rangle = 0$$

for all $n \geqslant 0$. Thus we have

$$\langle f^{-1}(t) | x^n \rangle = \left\langle \sum_{k=0}^{\infty} a_k t^k | x_n \right\rangle,$$

where we have simply dropped the term involving $t^{-1}$. If $g(t)$ is in $\mathscr{F}$ the expression $\langle g(t) f^{-1}(t) | x^n \rangle$ is evaluated by first taking the formal product $g(t) f^{-1}(t)$ before dropping any terms involving $t^{-1}$.

THEOREM 4.1. *Let $f(t)$ be a delta series and let $g(t)$ be any series in $\mathscr{F}$. Then*

$$\langle g(t) f^{-1}(t) | x^n \rangle = \langle g(t) | f^{-1}(t)\, x^n \rangle.$$

*Proof.* By linearity we need only observe that

$$\langle t^k | t^{-1} x^n \rangle = \langle t^{k-1} | x^n \rangle \qquad \text{for all } n, k \geqslant 0.$$

THEOREM 4.2. *Let $f(t)$ be a delta series and let $g(t)$ be any series in $\mathscr{F}$. Then as linear operators we have*

$$g(t) \circ f^{-1}(t) = g(t) f^{-1}(t),$$

*where the left side is composition of operators and the right side is the product of formal Laurent series.*

*Proof.* This follows readily from the fact that $t^k \circ t^{-1} x^n = t^{k-1} x^n$.

Notice that since the product is commutative we have

$$g(t) \circ f^{-1}(t) = g(t) f^{-1}(t) = f^{-1}(t)\, g(t).$$

Caution must be exercised, however, since $f^{-1}(t) \circ g(t) \neq f^{-1}(t)\, g(t)$.

We can now derive our first formula. Let $f(t)$ be a delta series and let $g(t)$ be invertible. For all integers $k \geqslant 0$ we define the linear operator $J_k$ on $P$ by

$$J_k x^n = f^{-1}(t; k)\, x^n - (1/c_0\, g_k)\langle g(t; k) f^{-1}(t; k) | x^n \rangle,$$

where $g_k$ is the constant term of $g(t; k)$.

THEOREM 4.3. *Let $s_n(x)$ be the decentralized Sheffer sequence for $(g(t), f(t))$. Then*

$$s_n(x) = (c_n/c_0\, g_n)\, J_0 \circ J_1 \circ \cdots \circ J_{n-1} 1,$$

*where by $J_0 \circ J_1 \circ \cdots \circ J_{n-1} 1$ we mean the composition of the operators $J_0, J_1, \ldots, J_{n-1}$ applied to the polynomial $p(x) = 1$. If $n = 0$, then $J_0 \circ \cdots \circ J_{n-1}$ is the identity operator. Again $g_n$ is the constant term in $g(t; n)$.*

*Proof.* The result is clear for $n = 0$. Since $\deg f(t; k) > 0$ we have

$$\begin{aligned} f(t; k) \circ J_k x^n &= f(t; k) \circ f^{-1}(t; k)\, x^n \\ &\quad - (1/c_0\, g_k)\langle g(t; k) f^{-1}(t; k) | x^n \rangle f(t; k)\, 1 \\ &= f(t; k) \circ f^{-1}(t; k)\, x^n \\ &= x^n \end{aligned}$$

and so

$$f(t; k) \circ J_k = I. \tag{4.1}$$

Also, since $\deg g(t; k) = 0$, Theorem 4.1 gives

$$\begin{aligned} \langle g(t; k) | J_k x^n > &= \langle g(t; k) | f^{-1}(t; k)\, x^n \rangle \\ &\quad - (1/c_0\, g_k)\langle g(t; k) f^{-1}(t; k) | x^n \rangle \langle g(t; k) | 1 \rangle \\ &= \langle g(t; k) | f^{-1}(t; k)\, x^n \rangle - \langle g(t; k) f^{-1}(t; k) | x \rangle \\ &= 0. \end{aligned} \tag{4.2}$$

Therefore, for $k \leqslant n$, Eq. (4.1) implies

$$\begin{aligned} &\langle g(t; k) f(t; 0) \cdots f(t; k-1) | (c_n/c_0\, g_n) J_0 \circ \cdots \circ J_{n-1} 1 \rangle \\ &\quad = (c_n/c_0\, g_n) \langle g(t; k) | f(t; k-1) \cdots f(t; 0) \circ J_0 \circ \cdots \circ J_{n-1} 1 \rangle \\ &\quad = (c_n/c_0\, g_n) \langle g(t; k) | J_k \circ \cdots \circ J_{n-1} 1 \rangle. \end{aligned} \tag{4.3}$$

If $k < n$, then (4.2) implies this equals 0 and if $k = n$ we get $(c_n/c_0\, g_n) \langle g(t; n) | 1 \rangle = c_n$. In either case we have $c_n \delta_{n,k}$. From degree considerations it follows that (4.3) equals $c_n \delta_{n,k}$ for all $n, k \geqslant 0$. This completes the proof.

We may derive another formula for decentralizated Sheffer sequences. Let $f(t)$ be a delta series and let $g(t)$ be invertible. We define the linear operators $H_k$ on $P$ by

$$H_0 = (g(t; 0))^{-1} t^{-1}$$

and for $k > 0$

$$H_k = (g(t; k-1)/g(t; k)) f^{-1}(t; k-1).$$

We denote the leading coefficient of the series $H_k$ by $h_k$.

THEOREM 4.4. *Let $s_n(x)$ be the decentralized Sheffer sequence for $(g(t), f(t))$. Then*

$$s_n(x) = (c_n/c_0)\, h_n H_0 \circ H_1 \circ \cdots \circ H_{n-1} 1,$$

*where if $n = 0$, then $s_0(x) = h_0$.*

*Proof.* Let us write $p_n(x) = (c_n/c_0)\, h_n H_0 \circ \cdots \circ H_{n-1} 1$ and verify the conditions of Theorem 3.3 for $p_n(x)$. Again, we denote the leading coefficient of $g(t; 0)$ by $g_0$ and the leading coefficient of $f(t; 0)$ by $f_0$. To prove part (1) of Theorem 3.3 we have for $n > 0$,

$$\begin{aligned}
\langle g(t; 0) \mid p_n(x)\rangle &= \langle g(t; 0) \mid (c_n/c_0)\, h_n H_0 \circ \cdots \circ H_{n-1} 1\rangle \\
&= \langle g(t; 0)\, H_0 \mid (c_n/c_0)\, h_n H_1 \circ \cdots \circ H_{n-1} 1\rangle \\
&= \langle t^{-1} \mid (c_n/c_0)\, h_n H_1 \circ \cdots \circ H_{n-1} 1\rangle = 0
\end{aligned}$$

and for $n = 0$,

$$\langle g(t; 0) \mid p_0(x)\rangle = \langle g(t; 0) \mid h_0\rangle = c_0\, g_0 h_0 = c_0.$$

To prove part (2) for $n = 1$ we have

$$\begin{aligned}
f(t; 0)\, p_1(x) &= f(t; 0)(c_n/c_0)\, h_1 H_0 1 \\
&= (c_1/c_0)\, h_1 (g(t; 0))^{-1} (f(t; 0)/t)\, 1 \\
&= (c_1/c_0)\, h_1 (f_0/g_0) \\
&= (c_1/c_0)\, S^1 p_0(x).
\end{aligned}$$

For $n > 1$ we first observe that

$$S^1 H_k = H_{k+1}$$

for $k \geqslant 1$ and by Theorem 4.2

$$
\begin{aligned}
f(t;0) \circ H_0 \circ H_1 &= f(t;0) \circ (g(t;0))^{-1}\, t^{-1} \circ (g(t;0)/g(t;1)) f^{-1}(t;1) \\
&= (g(t;0))^{-1}\, (f(t;0)/t) \circ (g(t;0)/g(t;1)) f^{-1}(t;1) \\
&= (g(t;1))^{-1}\, t^{-1} \\
&= S^1 H_0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(t;0)\, s_n(x) &= f(t;0)(c_n/c_0)\, h_n H_0 \circ H_1 \circ \cdots \circ H_{n-1} 1 \\
&= (c_n/c_{n-1})(c_{n-1}/c_0)\, S^1 h_{n-1} S^1 H_0 \circ S^1 H_1 \circ \cdots \circ S^1 H_{n-2} 1 \\
&= (c_n/c_{n-1})\, S^1 p_{n-1}(x).
\end{aligned}
$$

This completes the proof.

## 5. The Classical Umbral Calculus

The term *classical umbral calculus* will refer to the case

$$c_n = n!$$

for all $n \geqslant 0$. In this case the operator $t^k$ is the ordinary $k$th derivative and the linear functional $t^k$ is the $k$th derivative evaluated at 0.

The evaluation functional $\varepsilon_y(t)$ is the exponential series $e^{yt}$,

$$\langle e^{yt} \mid p(x) \rangle = p(y).$$

Therefore, the product of evaluation at $y$ with evaluation at $z$ is evaluation at $y + z$,

$$e^{yt} e^{zt} = e^{(y+z)t}.$$

The operator $e^{yt}$ is translation by $y$,

$$e^{yt} p(x) = p(x+y).$$

The expression $c_n/c_k c_{n-k}$ becomes the familiar binomial coefficient $\binom{n}{k}$ and the decentralizer Sheffer identity is

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y)\, g(t;k)\, s_{n-k}(x;k).$$

*The Gončarov Polynomials*

Let us take

$$g(t) = e^{y_0 t}$$

and

$$f(t) = t.$$

Then the decentralized geometric sequence for $(g(t), f(t))$ is

$$h_k(t) = e^{y_k t} t^k,$$

where

$$\langle h_k(t) \mid p(x) \rangle = p^{(k)}(y_k).$$

The decentralized Sheffer sequence for $(e^{y_0 t}, t)$ is the *Gončarov sequence.* We shall soon see that the $n$th Gončarov polynomial involves onlv the transcendentals $y_0, ..., y_{n-1}$ and we shall use the notation $G_n(x; y_0, ..., y_{n-1})$ for the $n$th Gončarov polynomial.

From the definition we have

$$G_n^{(k)}(y_k; y_0, ..., y_{n-1}) = n! \delta_{n,k}.$$

Theorem 3.3 characterizes the Gončarov polynomials by

$$\begin{aligned} G_n(y_0; y_0, ..., y_{n-1}) &= \delta_{n,0} \\ G_n'(x; y_0, ..., y_{n-1}) &= n G_{n-1}(x; y_1, ..., y_{n-1}). \end{aligned}$$

The decentralized Sheffer identity gives

$$\begin{aligned} &G_n(x + y; y_0, ..., y_{n-1}) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} G_k(y; y_0, ..., y_{k-1})\, G_{n-k}(x + y_k; y_k, ..., y_n) \end{aligned}$$

which is a new identity for Gončarov polynomials.

As a application of the expansion theorem we expand the polynomials $xG_{n-1}(x; y_1, ..., y_{n-1})$ in terms of $G_k(x; y_0, ..., y_{k-1})$. Corollary 1 of Theorem 3.2 gives

$$\begin{aligned} &xG_{n-1}(x; y_1, ..., y_{n-1}) \\ &\quad = \sum_{k=0}^{n} \frac{\langle e^{y_k t} t^k \mid xG_{n-1}(x; y_1, ..., y_{n-1}) \rangle}{k!} G_k(x; y_0, ..., y_{n-1}). \end{aligned}$$

But recalling that $\langle f(t)|xp(x)\rangle = \langle f'(t)|\,p(x)\rangle$ we have

$$\begin{aligned}
&\langle e^{y_k t}t^k|\,xG_{n-1}(x; y_1,..., y_{n-1})\rangle \\
&\quad = \langle (e^{y_k t}t^k)'\,|\,G_{n-1}(x; y_1,..., y_{n-1})\rangle \\
&\quad = \langle y_k e^{y_k t}t^k + ke^{y_k t}t^{k-1}|\,G_{n-1}(x; y_1,..., y_{n-1})\rangle \\
&\quad = y_k(n-1)_k\, G_{n-1-k}(y_k; y_{k+1},..., y_{n-1}) \\
&\qquad + k(n-1)_{k-1}\, G_{n-k}(y_k; y_k,..., y_{n-1}) \\
&\quad = y_k(n-1)_k\, G_{n-1-k}(y_k; y_{k+1},..., y_{n-1}) + n!\delta_{n,k}
\end{aligned}$$

and so

$$\begin{aligned}
xG_{n-1}(x; y_1,..., y_{n-1}) = G_n(x; y_0,..., y_{n-1}) + \sum_{k=0}^{n-1} \binom{n-1}{k} y_k \\
\times G_{n-1-k}(y_k; y_{k+1},..., y_{n-1})\, G_n(x; y_0,..., y_{k-1}).
\end{aligned}$$

This result is due to N. Levinson, for use in obtaining a bound for Whittaker's constant.

A formula for the Gončarov polynomials may be obtained from Theorem 4.3. In this case $f^{-1}(t; k) = t^{-1}$ and so

$$\begin{aligned}
J_k x^n &= t^{-1}x^n - \langle e^{y_k t}t^{-1}|\,x^n\rangle \\
&= (n+1)^{-1}\, x^{n+1} - \langle e^{y_k t}|\,t^{-1}x^n\rangle \\
&= (n+1)^{-1}\, x^{n+1} - (n+1)^{-1}\, y_k^{n+1} \\
&= \int_{y_k}^{x} t^n\, dt.
\end{aligned}$$

From this follows the usual formula

$$G_n(x; y_0,..., y_{n-1}) = n! \int_{y_0}^{x} dt_1 \int_{y_1}^{t_1} dt_2 \cdots \int_{y_{n-1}}^{t_{n-1}} dt_n.$$

If the difference operator $\Delta_y$ is defined by $\Delta_y p(x) = p(x) - p(y)$ and if we suggestively write $t^{-1}$ as $D^{-1}$ we have

$$J_k x^n = \Delta_{y_k} \circ D^{-1}x^n$$

and so $J_k = \Delta_{y_k} \circ D^{-1}$. Thus,

$$G_n(x; y_0,..., y_{n-1}) = n!\,\Delta_{y_0} \circ D^{-1} \circ \cdots \circ \Delta_{y_{n-1}} \circ D^{-1}\, 1.$$

Theorem 4.4 gives a different, and apparently new, formula for the Gončarov polynomials. We have

$$H_0 = e^{-y_0 t} t^{-1}$$

and for $k \geqslant 1$

$$H_k = e^{(y_{k-1} - y_k)t} t^{-1}.$$

Then

$$G_n(x; y_0, ..., y_{n-1}) = n! e^{-y_0 t} t^{-1} \circ e^{(y_0 - y_1)t} t^{-1} \circ \cdots \\ \circ e^{(y_{n-2} - y_{n-1})t} t^{-1}.$$

If we denote the translation operator $e^{zt}$ by $T_z$, then we obtain (with $t^{-1}$ written as $D^{-1}$)

$$G_n(x; y_0, ..., y_{n-1}) = n! T_{-y_0} \circ D^{-1} \circ T_{y_0 - y_1} \circ D^{-1} \circ \cdots \\ \circ T_{y_{n-2} - y_{n-1}} \circ D^{-1}.$$

A generalization of the Gončarov polynomials is obtained by taking $g(t) = e^{y_0 t}$ and $f(t)$ to be any series which is independent of the transcendentals. The decentralized Sheffer sequence for the pair $(e^{y_0 t}, f(t))$ satisfies

$$\langle e^{y_k t} f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k}.$$

In other words, $s_n(x)$ is the sequence of interpolation polynomials for the sequence of linear functionals $\varepsilon_{y_k}(t) \circ f(t)^k$. From Theorem 4.3 we have

$$J_k x^n = f^{-1}(t)\, x^n - \langle e^{y_k t} f^{-1}(t) \mid x^n \rangle \\ = \Delta_{y_k} \circ f^{-1}(D)\, x^n$$

and so

$$s_n(x) = n! \Delta_{y_0} \circ f^{-1}(D) \circ \cdots \circ \Delta_{y_{n-1}} \circ f^{-1}(D)\, 1.$$

From Theorem 4.4 we have the alternative formula

$$s_n(x) = (n!/f_1)\, T_{-y_0} \circ D^{-1} \circ T_{y_0 - y_1} \circ f^{-1}(D) \circ \cdots \circ T_{y_{n-2} - y_{n-1}} \circ f^{-1}(D)\, 1.$$

## 6. The Newtonian Umbral Calculus

The term *Newtonian umbral calculus* will refer to the case

$$c_n = 1$$

for all $n \geqslant 0$. The operator $t$ is multiplication by $x^{-1}$

$$tx^n = x^{n-1},$$

where $t1 = 0$.

The evaluation functional is the geometric series

$$\varepsilon_y(t) = \sum_{k=0}^{\infty} y^k t^k = \frac{1}{1-yt}$$

and so

$$\frac{1}{1-yt} \,|\, p(x)\rangle = p(y).$$

The operator $\varepsilon_y(t) = (1-yt)^{-1}$ satisfies

$$\varepsilon_y(t)\, x^n = \frac{1}{1-yt} x^n = \sum_{k=0}^{\infty} y^k t^k x^n = \sum_{k=0}^{n} y^k x^{n-k} = \frac{x^{n-1} - y^{n+1}}{x-y}$$

and so

$$\frac{1}{1-yt} p(x) = \frac{xp(x) - yp(y)}{x-y}.$$

Thus, the decentralized-Sheffer identity becomes

$$\frac{xs_n(x) - ys_n(y)}{x-y} = \sum_{k=0}^{n} s_k(y)\, g(t; k)\, s_{n-k}(x; k).$$

Let us compute the decentralized Sheffer sequence for $((1-y_0 t)^{-1}, t)$. From Theorem 4.3 we obtain in the usual way

$$s_n(x; y_0, \ldots, y_{n-1}) = \Delta_{y_0} \circ X \circ \cdots \circ \Delta_{y_{k-1}} \circ X1,$$

where $X$ is the operator multiplication by $x$. This sequence is the Newtonian analog of the Gončarov sequence. It satisfies

$$\langle \varepsilon_{y_k}(t) | X^{-k} s_n(x; y_0, \ldots, y_{n-1}) \rangle = \delta_{n,k}$$

and

$$X^{-1} s_n(x; y_0, \ldots, y_{n-1}) = s_{n-1}(x; y_1, \ldots, y_{n-1}),$$

where of course $X^{-k} x^j = 0$ if $j - k < 0$.

Notice that the delta series $t\varepsilon_y(t) = t/(1-yt)$ satisfies

$$\frac{t}{1-yt}x^n = \frac{1}{1-yt}x^{n-1} = \frac{x^n - y^n}{x-y}$$

and so

$$(t/(1-y^t))\,p(x) = (p(x)-p(y))/(x-y) \tag{6.1}$$

which is the first divided difference of $p(x)$.

The $k$th *divided difference* of a polynomial $p(x)$ with respect to a sequence $y_n$ is usually defined by a recurrence (see, e.g., Davis [4] or Hildebrand [5])

$$p[y_0] = p(y_0)$$
$$p[y_0, y_1] = (p[y_0] - p[y_1])/(y_0 - y_1)$$
$$p[y_0,..., y_k] = (p[y_0,..., y_{k-1}] - p[y_1,..., y_k])/(y_0 - y_k).$$

We are now in a position to describe these divided differences directly. We begin by observing that

$$p[y_0] = \left\langle \frac{1}{1-yt} \,\middle|\, p(x) \right\rangle$$

and

$$p[y_0, y_1] = ((1-y_0t)^{-1} - (1-y_1t)^{-1})/(y_0-y_1)\,|\, p(x)\rangle$$
$$= \left\langle \frac{t}{(1-y_0t)(1-y_1t)} \,\middle|\, p(x) \right\rangle$$

and

$$p[y_0, y_1, y_2] = \left\langle \frac{(t/(1-y_0t)(1-y_1t)) - (t/(1-y_1t)(1-y_2t))}{y_0 - y_2} \,\middle|\, p(x) \right\rangle$$
$$= \left\langle \frac{t^2}{(1-y_0t)(1-y_1t)(1-y_2t)} \,\middle|\, p(x) \right\rangle.$$

The pattern is now clear. The decentralized geometric sequence

$$h_k(t) = \varepsilon_{y_k}(t)\, t\varepsilon_{y_0}(t) \cdots t\varepsilon_{y_{k-1}}(t) = t^k/(1-y_0t)\cdots(1-y_kt)$$

has the property that

$$h_{k+1}(t) = (h_k(t) - S^1 h_k(t))/(y_0 - y_{k+1})$$

and since $\langle h_0(t) \,|\, p(x)\rangle = p[y_0]$ we deduce that

$$\left\langle \frac{t^k}{(1-y_0t)\cdots(1-y_kt)} \,\middle|\, p(x) \right\rangle = p[y_0,..., y_k].$$

Notice that $h_k(t)$ is the decentralized geometric sequence for $g(t)=(1-y_0 t)^{-1}=\varepsilon_{y_0}(t)$ and $f(t)=t/(1-y_0 t)=t\varepsilon_{y_0}(t)$.

Naturally we want next to determine the decentralized Sheffer sequence for $((1-y_0 t)^{-1}, t/(1-y_0 t))$. In order to use Theorem 4.4 we compute

$$H_0=(1-y_0 t)\,t^{-1}=t^{-1}-y_0=X-y_0,$$

where $X=t^{-1}$ is the operator multiplication by $X$. Also, for $k>0$,

$$H_k=\frac{1-y_k t}{1-y_{k-1}t}\,\frac{1-y_{k-1}t}{t}=t^{-1}-y_k t=X-y_k.$$

Thus the decentralized Sheffer sequence for $((1-y_0 t)^{-1}, t/(1-y_0 t))$ is

$$\pi_n(x; y_0,..., y_{n-1})=(x-y_0)(x-y_1)\cdots(x-y_{n-1}).$$

Thus, from the definition of decentralized Sheffer sequence

$$\pi_n[y_0,..., y_k]=\delta_{n,k}.$$

Theorem 3.3 characterizes $\pi_n(x; y_0,..., y_{n-1})$ as the unique polynomial sequence for which

$$\pi_n(y_0,..., y_{n-1})=\delta_{n,0}$$

and

$$\begin{aligned}&(\pi_n(x; y_0,..., y_{n-1})-\pi_n(y_0; y_0,..., y_{n-1}))/(x-y_0)\\&\quad=\pi_{n-1}(x; y_1,..., y_{n-1}).\end{aligned}$$

It is interesting to notice that

$$\begin{aligned}\left\langle \frac{t^k}{(1-yt)^{k+1}}\,\middle|\, x^n\right\rangle &= \sum_{j=0}^{\infty}\binom{j+k}{j} y^j\langle t^{k+j}\,|\,x^n\rangle\\ &=\binom{n}{k}y^{n-k}\\ &=\frac{1}{k!}\langle \varepsilon_y(t)\,|\,D^k x^n\rangle\end{aligned}$$

and so $t^k/(1-yt)^{k+1}$ is $1/k!$ times the $k$th derivative evaluated at $y$. We may use this result to handle the case of confluent transcendentals. In particular suppose $y_0=y_1=\cdots=y_j$ for $j<k$. Then using (6.1) we have

$$\begin{aligned}p[y_0, y_1,..., y_{k-1}] &= \left\langle \frac{t^{k+1}}{(1-y_0 t)\cdots(1-y_{k-1}t)}\,\middle|\, p(x)\right\rangle\\ &=\left\langle \frac{t^j}{(1-y_0 t)^{j+1}}\,\middle|\,\frac{t^{k-j-1}}{(1-y_{j+1}t)\cdots(1-y_{k-1}t)}\,p(x)\right\rangle\\ &=(1/j!)\langle \varepsilon_{y_0}(t)\,|\,D^j p[x, y_{j+1},..., y_{k-1}]\rangle\end{aligned}$$

It is worth taking special note of the case of two transcendentals,

$$\frac{p(y)-p(z)}{y-z}=\left\langle \frac{t}{(1-yt)(1-zt)} \,\middle|\, p(x) \right\rangle.$$

Now we would like to set $y=z$, which can be done in the right side and gives

$$\left\langle \frac{t}{(1-yt)^2} \,\middle|\, p(x) \right\rangle = \langle \varepsilon_y(t) \mid Dp(x) \rangle.$$

Thus, we haved passed from the difference quotient to the derivative in a purely algebraic manner, requiring no use of the limit! This brings up many interesting questions about a purely algebraic polynomials calculus, without the need for the concept of limit. We plan to continue this story at a future date.

## 7. Formal Power Series

Up to now the umbral calculus has taken the point of view that the dual-vector space $P^*$ can be profitably studied via the algebra $\mathscr{F}$ of formal power series in a single variable. We shall now take the reverse point of view. Since any linear functional on $P$ is a formal power series—so any formal power series is a linear functional. In fact, a formal power series is a linear functional in many ways—one for each choice of the sequence $c_n$.

We shall confine our attention here to one problem. A more detailed study of the algebra of formal-power series awaits a future paper in this series. Let $c_n$ and $d_n$ be sequences of nonzero constants. Let $g(t)$ and $h(t)$ be invertible series in $\mathscr{F}$ and let $f(t)$ and $l(t)$ be delta series. Consider the formal power series

$$u(t)=\sum_{k=0}^{\infty} \frac{a_k}{c_k} g(t;k) f(t;0) \cdots f(t;k-1)$$

for constants $a_k$ in $K$. Then $u(t)$ has an expression of the form

$$u(t)=\sum_{k=0}^{\infty} \frac{b_k}{d_k} h(t;k)\, l(t;0) \cdots l(t;k-1)$$

for some constants $b_k$. We wish to determine the constants $b_k$ in terms of the constants $a_k$.

A word of notation is in order. Since we are considering more than one sequence of nonzero constants, we have more than one action of $\mathscr{F}$ on $P$. We shall use the notation

$$\langle f(t) \mid p(x) \rangle_c$$

to denote the action of the linear functional $f(t)$ on $p(x)$ derived from

$$\langle t^k | x^n \rangle = c_n \delta_{n,k}.$$

Similarly, we shall write $\langle f(t) | p(x) \rangle_d$ for the action of $f(t)$ on $p(x)$ derived from $\langle t^k | x^n \rangle = d_n \delta_{n,k}$. The following lemma will be of use:

LEMMA 7.1. *If $u(t)$ is in $\mathscr{F}$, then*

$$\langle u(t) | x^n \rangle_d / d_n = \langle u(t) | x^n \rangle_c / c_n.$$

*Proof.* This result is obvious for $u(t) = t^k$ and an appeal to linearity completes the proof.

THEOREM 7.1. *Let $u(t)$ be the formal series*

$$u(t) = \sum_{k=0}^{\infty} \frac{a_k}{c_k} g(t; k) f(t; 0) \cdots f(t; k-1), \tag{7.1}$$

*where $a_k$ is in $K$. Then*

$$u(t) = \sum_{k=0}^{\infty} \frac{b_k}{d_k} h(t; k)\, l(t; 0) \cdots l(t; k-1), \tag{7.2}$$

*where*

$$b_n = \sum_{j=0}^{n} \frac{\langle u(t) | x^j \rangle_c}{c_j} \langle t^j | r_n(x) \rangle_d,$$

*where $r_n(x)$ is the decentralized Sheffer sequence for $(h(t), l(t))$ using the sequence of constants $d_n$. That is,*

$$\langle h(t; k)\, l(t; 0) \cdots l(t; k-1) | r_n(x) \rangle_d = d_n \delta_{n,k}.$$

*Proof.* It is clear from degree consideration that an expression of the form (7.2) exists. Applying both sides of (7.2) to $r_n(x)$, using the sequence $d_n$, gives

$$b_n = \langle u(t) | r_n(x) \rangle_d.$$

By Corollary 3.1 we have

$$r_n(x) = \sum_{j=0}^{n} \frac{\langle t^j | r_n(x) \rangle_d}{d_j} x^j$$

and thus,

$$b_n = \sum_{j=0}^{n} \frac{\langle t^j | r_n(x) \rangle_d}{d_j} \langle u(t) | x^j >_d.$$

An application of Lemma 7.1 concludes the proof.

When we choose to take $c_n = d_n$ this result can be simplified.

THEOREM 7.2. *Let $u(t)$ be the formal series*

$$u(t) = \sum_{k=0}^{\infty} \frac{a_k}{c_k} g(t; k) f(t; 0) \cdots f(t; k-1),$$

*where $a_k$ is in $K$. Then*

$$u(t) = \sum_{k=0}^{\infty} \frac{b_k}{c_k} h(t; k)\, l(t; 0) \cdots l(t; k-1),$$

*where*

$$b_n = \sum_{j=0}^{n} \frac{a_j}{c_j} \langle g(t;j) f(t; 0) \cdots f(t; j-1) | r_n(x) \rangle,$$

*where $r_n(x)$ is the decentralized Sheffer sequence for $(h(t), l(t))$.*

*Proof.* Let $s_n(x)$ be the decentralized Sheffer sequence for $(g(t), f(t))$. Then Corollary 3.1 gives

$$r_n(x) = \sum_{j=0}^{n} \frac{s_j(x)}{c_j} \langle g(t;j) f(t; 0) \cdots f(t; j-1) | r_n(x) \rangle.$$

Applying $u(t)$ to both sides and observing that

$$\langle u(t) | r_n(x) \rangle = b_n$$

and

$$\langle u(t) | s_j(x) \rangle = a_j$$

concludes the proof.

As a special case we have

COROLLARY 7.1. *Let*

$$u(t) = \sum_{k=0}^{\infty} \frac{a_k}{c_k} t^k.$$

*Then*

$$u(t)=\sum_{k=0}^{\infty}\frac{b_k}{c_k}h(t;k)\,l(t;0)\cdots l(t;k-1),$$

*where*

$$b_n=\sum_{j=0}^{n}\frac{a_j}{c_j}\langle t^j\,|\,r_n(x)\rangle,$$

*where* $r_n(x)$ *is the decentralized Sheffer sequence for* $(h(t),l(t))$.

Let us give some examples.

EXAMPLE 1. Let

$$u(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k.$$

We determine $b_n$ in

$$u(t)=\sum_{k=0}^{\infty}\frac{b_k}{k!}e^{y_k t}t^k.$$

Referring to Corollary 7.1 we have $c_n=n!$ and

$$h(t)=e^{y_0 t},\qquad l(t)=t$$

and so

$$r_n(x)=G_n(x;y_0,\ldots,y_{n-1})$$

is the Gončarov sequence. Thus,

$$\begin{aligned}b_n&=\sum_{j=0}^{n}\frac{a_j}{j!}\langle t^j\,|\,G_n(x;y_0,\ldots,y_{n-1})\rangle\\&=\sum_{j=0}^{n}\binom{n}{j}a_jG_{n-j}(0;y_j,\ldots,y_{n-1})\end{aligned}$$

and we may write

$$\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\sum_{j=0}^{k}\binom{k}{j}a_jG_{n-j}(0;y_j,\ldots,y_{n-1})\right]e^{y_k t}t^k.$$

EXAMPLE 2. Let

$$u(t) = \sum_{k=0}^{\infty} a_k t^k.$$

We determine $b_n$ in

$$u(t) = \sum_{k=0}^{\infty} b_k \frac{t^k}{(1 - y_0 t) \cdots (1 - y_k t)}.$$

Referring to Corollary 7.1 we have $c_n = 1$ and

$$h(t) = \frac{1}{1 - y_0 t}, \qquad l(t) = \frac{t}{1 - y_0 t}$$

and so

$$r_n(x) = (x - y_0) \cdots (x - y_{n-1}).$$

Thus,

$$b_n = \sum_{j=0}^{n} a_j \langle t^j \mid (x - y_0) \cdots (x - y_{n-1}) \rangle.$$

If we write $\sigma(n, j)$ for the elementary symmetric function on $y_0, \ldots, y_{n-1}$ of order $j$, then

$$(x - y_0) \cdots (x - y_{n-1}) = \sum_{i=0}^{n} (-1)^k \sigma(n; i)\, x^{n-i}$$

and so

$$\langle t^j \mid (x - y_0) \cdots (x - y_{n-1}) \rangle = (-1)^{n-j} \sigma(n, n-j).$$

Therefore,

$$b_n = \sum_{j=0}^{n} (-1)^{n-j} a_j \sigma(n, n-j)$$

and

$$\sum_{k=0}^{\infty} a_k A^k = \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k} (-1)^{k-k} a_j \sigma(n, n-j) \right] \times \frac{t^k}{(1 - y_0 t) \cdots (1 - y_k t)}.$$

EXAMPLE 3. Let

$$u(t)=\sum_{k=0}^{\infty} \frac{a_k}{k!} e^{y_k t} t^k.$$

We determine $b_n$ in

$$u(t)=\sum_{k=0}^{\infty} b_k \frac{t^k}{(1-z_0 t) \cdots (1-z_k t)}.$$

Referring to Theorem 7.1 we have

$$c_n=n!, \qquad g(t)=e^{y_0 t}, \qquad f(t)=t,$$

and

$$d_n=1, \qquad h(t)=(1-z_0 t)^{-1}, \qquad l(t)=t/(1-z_0 t),$$

and so

$$r_n(x)=(x-z_0) \cdots (x-z_{n-1})=\sum_{i=0}^{n} (-1)^k \sigma(n, i)\, x^{n-i}. \tag{7.3}$$

Therefore

$$b_n=\sum_{j=0}^{n} \frac{\langle u(t) | x^j \rangle_c}{j!} \langle t^j | (x-z_0) \cdots (x-z_{n-1}) \rangle_d.$$

We see that

$$\langle t^j | (x-z_0) \cdots (x-z_{n-1}) \rangle_d=(-1)^{n-j} \sigma(n, n-j)$$

and

$$\begin{aligned} \langle u(t) | x^j \rangle_c &= \sum_{i=0}^{\infty} \frac{a_i}{i!} \langle e^{y_i t} t^k | x^j \rangle_c \\ &= \sum_{i=0}^{j} \binom{j}{i} a_i y_i^{j-i}. \end{aligned}$$

Hence,

$$b_n=\sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(-1)^{n-j}}{j!} \binom{j}{i} a_i y_i^{j-i} \sigma(n, n-j).$$

EXAMPLE 4. Let

$$u(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{(1-y_0 t) \cdots (1-y_k t)}.$$

We determine $b_n$ in

$$u(t) = \sum_{k=0}^{\infty} \frac{b_k}{k!} e^{z_k t} t^k.$$

Referring to Theorem 7.1 we have

$$c_n = 1, \qquad g(t) = (1-y_0 t)^{-1}, \qquad f(t) = t/(1-y_0 t),$$

and

$$d_n = n!, \qquad h(t) = e^{z_0 t}, \qquad l(t) = t.$$

Therefore,

$$r_n(x) = G_n(x; z_0, ..., z_{n-1})$$

and

$$b_n = \sum_{j=0}^{n} \langle u(t) | x^j \rangle_c \langle t^j | G_n(x; t_0, ..., z_{n-1}) \rangle_d.$$

Now

$$\langle t^j | G_n(x; z_0, ..., z_{n-1}) \rangle_d = (n)_j G_{n-j}(0; z_j, ..., z_{n-1})$$

and

$$\langle y(t) | x^i \rangle_c = \sum_{i=0}^{j} a_i \langle t^i/(1-y_0 t) \cdots (1-y_i t) | x^i \rangle.$$

If we use the notation $[y_0, ..., y_k]_{x^j}$ for the $k$th divided difference of $x^j$, then

$$\langle u(t) | x^j \rangle_c = \sum_{i=0}^{j} a_i [y_0, ..., y_i]_{x^j}$$

and so

$$b_n = \sum_{j=0}^{n} \sum_{i=0}^{j} (n)_j a_i [y_0, ..., y_i]_{x^j} G_{n-j}(0; z_j, ..., z_{n-1}).$$

## References

The first two references have extensive bibliographies and we shall not repeat them.

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