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# A PROBLEM ON MULTI-COLORINGS OF GRAPHS

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In this paper we consider colorings of the edges of the complete graph $K_m$ with $n$ colors such that the edges of any color form a non-trivial complete subgraph of $K_m$. We allow an edge of $K_m$ to have more than one color. Such a coloring will be called $r$-admissible if no cycle of length $r$ has a different color for each edge. Let $E(m,n,r)$ be the maximum number of incidences of colors and edges, taken over all $r$-admissible colorings of $K_m$ with $n$ colors. Then for $r=3,4$, and 5 we give an upper bound for $E(m,n,r)$; as well as a lower bound for $E(m,n,r)$ for all $r$. An analogue to a problem of Zarankiewicz concerning 0, 1-matrices is mentioned.

## 1. Introduction

A well known problem of Zarankiewicz [1, 2] is to determine the largest positive integer $M(m,n)$ for which there is an $m$ by $n$ matrix of zeros and ones with $M(m,n)$ ones and no 2 by 2 submatrix consisting entirely of ones. Suppose we modify this problem by asking instead for the largest positive integer $N(m,n)$ for which there is an $m$ by $n$ matrix of zeros and ones with $N(m,n)$ ones and no "hexagon" consisting entirely of ones, rather than no "square". By this we mean that given any three rows $r_1, r_2, r_3$ and any three columns $c_1, c_2, c_3$ of a matrix, not all of the entries $(r_1, c_1)$, $(r_1, c_2)$, $(r_2, c_2)$, $(r_2, c_3)$, $(r_3, c_1)$, and $(r_3, c_3)$ are ones. Now we identify the columns of the matrix with the colors $c_1, \ldots, c_n$; the rows with the vertices $v_1, \ldots, v_m$ of the complete graph $K_m$ and we interpret a pair of ones in the $i^{\text{th}}$ column as indicating that the two associated vertices form an edge of color $c_i$. We see that the $i^{\text{th}}$ column represents a complete subgraph of $K_m$ colored with color $c_i$. The "hexagon" described above is nothing more than a cycle of length $r=3$ whose edges are different colors. If $N_i$ is the number of ones in the $i^{\text{th}}$ column, then $\binom{N_i}{2}$ is the number of edges of color $c_i$. In searching for an upper bound for $N(m,n) = \sum N_i$, we may naturally look for an upper bound for $\sum \binom{N_i}{2}$, which is the number of incidences of edges of $K_m$ and colors. Therefore, we address the following problem suggested by this interpretation.

Let $K_m$ be the complete graph on $m$ vertices, and let $c_1, \ldots, c_n$ be $n$ distinct labels or colors. For each $i = 1, \ldots, n$ color some non-trivial complete subgraph of $K_m$ with color $c_i$. We permit an edge of $K_m$ to be colored with more than one color. Such a coloring will be called a multi-coloring of $K_m$. A multi-coloring will be called

$r$-admissible if there is no cycle of length $r$ each edge of which has a different color. We denote by $E(m, n, r)$ the maximum, taken over all $r$-admissible colorings, of the number of incidences of edges of $K_m$ and colors. We will show that

$$2\binom{m-1}{2}+n-1 \leqslant E(m, n, 3)<2\binom{m}{2}+n,$$

$$3\binom{m-2}{2}+3n-6 \leqslant E(m, n, 4)<3\binom{m}{2}+3n,$$

$$4\binom{m-3}{2}+6n-18 \leqslant E(m, n, 5)<4\binom{m}{2}+6n.$$

The above lower bounds are special cases of a more general inequality, namely

$$(r-1)\binom{m-r+2}{2}-\frac{(r-1)(r-2)^2}{2}+\binom{r-1}{2}n \leqslant E(m, n, r),$$

which we will also establish. If $n<r$ or $m<r$, then clearly any coloring is $r$-admissible and so $E(m, n, r)=n\binom{m}{2}$. In the sequel, we assume $n, m \geqslant r$.

The implication of these results to the aforementioned problem concerning "hexagons" is that:

$$\sum\binom{N_i}{2} \equiv E(m, n, 3)<2\binom{m}{2}+n.$$

For any integer $p \geqslant 1$, this leads (by rearranging the inequality $0 \leqslant (1/2p)(N_i-p-1)(N_i-p)$ and summing on $i$) to

$$N(m, n)=\sum N_i<\frac{1}{p} 2\binom{m}{2}+\left(\frac{1}{p}+\frac{p+1}{2}\right) n.$$

It is possible to obtain similar "matrix" interpretations for other values of $r$.

In the next sections, we establish the upper bound for $r=3,4$ and 5. Then we establish the lower bound for arbitrary $r$, and finally we mention some related problems.

## 2. The case $r=3$

**Theorem 2.1.** $E(m, n, 3)<2\binom{m}{2}+n$.

**Proof:** Suppose we are given a 3-admissible multi-coloring of $K_m$ with $E(m, n, 3)$ number of incidences of edges and colors. For each $i=1, \ldots, n$ let $F_i$ be the subgraph of $K_m$ whose edges are those edges of $K_m$ that are colored with $c_i$ and at least two other colors. Let the number of edges in $F_i$ be $f_i$. By the definition of multi-coloring, every edge of $K_m$ whose endpoints are in the vertex set of $F_i$ has color $c_i$. Therefore, 3-admissibility implies that $F_i$ can contain no path of length two.

Moreover, 3-admissibility implies that any edge of $K_m$ not in $F_i$, but whose endpoints are in the vertex set of $F_i$, can be colored with no other color except $c_i$. It follows that if $S$ is the number of edges of $K_m$ with exactly one color, we have

$$\sum \binom{f_i}{2} \leq \tfrac{1}{4} S$$

and so

$$\sum f_i \leq \tfrac{1}{4} S + n.$$

But $\Sigma f_i$ is the number of incidences of colors and edges of $K_m$ which have at least three colors, so if $U$ is the number of edges of $K_m$ with exactly two colors, we have

$$\begin{aligned} E(m, n, 3) &\leq \sum f_i + S + 2U \leq 2(S + U) + n \\ &\leq 2\binom{m}{2} + n \end{aligned}$$

Notice, however, that the last inequality will be strict unless $S + U = \binom{m}{2}$, in which case $f_i = 0$, for all $i$ and so

$$E(m, n, 3) \leq S + 2U < 2\binom{m}{2} + n.$$

## 3. The case $r = 4$

**Theorem 3.1.** $E(m, n, 4) < 3\binom{m}{2} + 3n$

**Proof:** We proceed as in the proof of Theorem 2.1. Given a 4-admissible multi-coloring of $K_m$ with $E(m, n, 4)$ number of incidences of colors and edges, we define $F_i$ to be the subgraph of $K_m$ with each edge having at least four colors, including $c_i$. As before, $f_i$ is the number of edges of $F_i$. We will show that unless $f_i \leq 3$, there are at least $\tfrac{1}{2} f_i$ edges of $K_m$ which are monochromatic of color $c_i$.

Here $F_i$ can have no open path of length three, and so the non-trivial connected components of $F_i$ are either stars (including 1-stars or edges) or triangles. Furthermore, every edge of $K_m$, not in $F_i$, but whose endpoints are in $F_i$, with the exception of those edges completing a triangle with a 2-star in $F_i$, must be monochromatic of color $c_i$. Therefore, if $F_i$ contains $t_i$ stars, $p_i$ 2-stars, and hence $f_i + t_i$ vertices, we must have at least $\binom{f_i + t_i}{2} - f_i - p_i$ edges which are monochromatic of color $c_i$. For $f_i \geq 3$, this is at least $\tfrac{1}{2} f_i$. So letting $S_i$ be the number of edges monochromatic of color $c_i$, we have $f_i \leq 2S_i + 3$. This gives us, with $S$ being the number of monochromatic edges in $K_m$,

$$\sum f_i \leq 2S + 3n.$$

Then if $B$ and $T$ are the number of bichromatic and trichromatic edges in $K_m$, we get

$$E(m, n, 4) \leqslant \Sigma f_i + S + 2B + 3T$$
$$\leqslant 3S + 2B + 3T + 3n$$
$$\leqslant 3\binom{m}{2} + 3n.$$

As before, strict inequality holds in the last inequality unless $f_i = 0$ for all $i$, in which case

$$E(m, n, 4) \leqslant S + 2D + 3T < 3\binom{m}{2} + 3n.$$

## 4. The case $r = 5$

**Theorem 4.1.** $E(m, n, 5) < 4\binom{m}{2} + 6n$

**Proof:** We proceed in a manner similar to the proofs of the previous theorems. Given a 5-admissible multi-coloring of $K_m$ with $E(m, n, 5)$ number of incidences of colors and edges, we define $F_i$ to be the subgraph of $K_m$ consisting of those edges having at least five colors, including $c_i$. Let $f_i$ be the number of edges in $F_i$. Also; let $B_i$ be the number of edges of $K_m$ with endpoints in $F_i$, colored with $c_i$ and at most one other color. We will show that $B_i \geqslant f_i - 6$.

We first notice that $F_i$ can have no open path of length four. This means that the non-trivial connected components of $F_i$ are of the following five types: (i) a $k$-star for $k \geqslant 1$, (ii) two stars whose centers are connected by an additional edge, (iii) a triangle, (iv) a triangle with some additional edges all connected to the same vertex of the triangle, and (v) a subgraph of $K_4$. Moreover, there is no open path of length four in $K_m$ such that all of its vertices are in $F_i$, two or more of its edges are in $F_i$, and the remaining edges have at least three colors.

Consider first the case that all the components of $F_i$ are 1-stars (edges). By the above remarks, it is not possible for a 1-star in $F_i$ to be connected to more than one other 1-star in $F_i$ by edges with at least three colors. Therefore

$$B_i \geqslant 4\binom{f_i}{2} - 4\left[\frac{f_i}{2}\right] \geqslant f_i - 6.$$

For the case that not all of the components of $F_i$ are 1-stars, we proceed by induction on the number of components, say $q$. If $q = 1$, noticing that the result is trivial when $f_i \leqslant 6$, it is routine to check all possible configurations for $F_i$ and verify $B_i \leqslant f_i - 6$. For when $F_i$ is a $k$-star, $k \geqslant 7$, all edges in $K_m - F_i$ whose endpoints are in $F_i$, with at most one exception, can have at most one color excluding $c_i$. When $F_i$ is a triangle with additional edges or two stars whose centers are connected no edge in $K_m - F_i$ whose endpoints are in $F_i$ can have more than one color, excluding $c_i$. These are the only possibilities for $F_i$ whenever $f_i \geqslant 7$. Assume the result is true for any $F_i$ with less than $q$ components. If $F_i$ has $q \geqslant 2$ components, let $G$ be a component with the minimum number of vertices, say $V(G)$. By the induction hypothesis, there are at least $f_i - E(G) - 6$ edges of $K_m$, whose endpoints lie in $F_i - G$, and which have color $c_i$ and at most one other color. Now there is some component $H$ of $F_i - G$ which contains an open path of length two. Therefore every edge of $K_m$ connecting a vertex of $G$ with an endpoint of this path is colored with $c_i$ and at most one other color. There are $2V(G)$ such edges, and for any component of $F_i$, $2V(G) \geqslant E(G)$. Hence

$$B_i \geqslant f_i - E(G) - 6 + 2V(G) \geqslant f_i - 6.$$

To complete the proof of the theorem, we let $B$ equal the number of edges of $K_m$ colored with at most two colors, we let $T$ be the number of edges of $K_m$ with exactly three colors, and $D$ be the number of edges with exactly four colors. Then

$$\begin{aligned} E(m, n, 5) &\leqslant \sum f_i + 2B + 3T + 4D \\ &\leqslant 3B + 3T + 4D + 6n \\ &\leqslant 4\binom{m}{2} + 6n. \end{aligned}$$

The last inequality will be strict unless $f_i = 0$ for all $i$, in which case

$$E(m, n, 5) \leqslant 2B + 3T + 4D < 4\binom{m}{2} + 6n.$$

## 5. The lower bounds

We will establish a lower bound for $E(m, n, r)$ by construction. Let $K_m$ be the complete graph on the points $\{v_1, \ldots, v_m\}$. For $m \geqslant 2r - 2$ we proceed as follows. Color all edges of $K_m$ with $c_1$. Color the complete graph on $\{v_1, \ldots, v_{m-r+1}\}$ with colors $c_2, \ldots, c_{r-1}$. Color the complete $r - 1$ graph on $\{v_{m-r+2}, \ldots, v_m\}$ with the remaining colors $c_r, \ldots, c_n$. It is easy to see that there is no inadmissible cycle of length $r$. Therefore

$$\begin{aligned} E(m, n, r) &\geqslant \binom{m}{2} + (r-2)\binom{m-r+1}{2} + \binom{r-1}{2}(n-r+1) \\ &= (r-1)\binom{m-r+1}{2} - \frac{(r-2)^2(r-1)}{2} + \binom{r-1}{2}n. \end{aligned}$$

For $r \leqslant m \leqslant 2r - 3$, color all edges of $K_m$ with color $c_1$. Color the complete graph on $\{v_1, \ldots, v_{r-1}\}$ with all the remaining colors. Then

$$E(m, n, r) \geqslant \binom{m}{2} + (n-1)\binom{r-1}{2}.$$

The author has so far been unable to find any $r$-admissible multi-colorings which would imply that $E(m, n, r)$ is greater than this lower bound. However, if we remove the requirement that all the colors be used in the coloring we can improve, in some instances, on this lower bound. In particular, if we use exactly $r-1$ colors we can color each edge of $K_m$ with each color, and not produce an inadmissible cycle of length $r$. Hence $E(m, n, r) \geq (r-1)\binom{m}{2}$ and this will be greater than the first lower bound for $m$ sufficiently large compared to $n$.

This, by the way, indicates the reason for having the term $(r-1)\binom{m}{2}$ in the upper bounds of Theorems 2.1, 3.1 and 4.1. The proofs of these theorems do not require that all of the $n$ colors be used in the multi-coloring. So we see from the above considerations that the right hand side of the inequalities must be at least $(r-1)\binom{m}{2}$.

## 6. Concluding remarks

Finally, we wish to mention some problems which are similar to the problem of this paper. One such problem arises if we change the definition of $r$-admissible to exclude the possibility of finding $r$ points of $K_m$ for which a different color can be chosen for each edge of the complete graph on these $r$ points.

Another variation arises if we replace $K_m$ by the complete bipartite graph $K(a, b)$ on two sets $V_a$ and $V_b$ and if we color some complete bipartite subgraph of $K(a, b)$ with color $c_i$ for $i = 1, \ldots, n$. Then we may define an $r, s$-admissible multi-coloring to be one in which there is no set of $r$ points from $V_a$ and $s$ points of $V_b$ for which we can choose a different color for each edge of $K(r, s)$. While this problem does seem similar to the problem of this paper, the solution seems to be of a different nature. For instance, consider the complete bipartite graph $K(a, b)$ with $a = 2$. Choosing $v$ in $V_a$, we color every edge of the complete bipartite subgraph $K(1, b)$ on $\{v\}$ and $V_b$ with each color $c_i$ for $i = 1, \ldots, n$. We therefore have a $2, k$-admissible multi-coloring for any $k$ satisfying $2 \leq k \leq b$. Furthermore, the number of incidences of colors and edges of $K(2, b)$ is $nb$, which is one half of the maximum possible number of such incidences in any multi-coloring. So we see that there are multi-colorings of $K(a, b)$ with the property that the number of incidences is a constant times the product of the number of colors and the number of edges of $K(a, b)$. This will produce results of a rather different nature than the ones in Theorems 2.1, 3.1 and 4.1. It is therefore not surprising that the methods used in the proofs in these theorems do not seem to be effective for this problem.

At the present time, the author knows of no effective method which will deal with either of these problems.

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## References

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