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A PROBLEM ON MULTI-COLORINGS OF GRAPHS

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In this paper we consider colorings of the edges of the complete graph K_m with n colors such that the edges of any color form a non-trivial complete subgraph of K_m. We allow an edge of K_m to have more than one color. Such a coloring will be called r-admissible if no cycle of length r has a different color for each edge. Let $E(m, n, r)$ be the maximum number of incidences of colors and edges, taken over all r-admissible colorings of K_m with n colors. Then for $r = 3, 4$, and 5 we give an upper bound for $E(m, n, r)$; as well as a lower bound for $E(m, n, r)$ for all r. An analogue to a problem of Zarankiewicz concerning 0, 1-matrices is mentioned.

1. Introduction

A well known problem of Zarankiewicz [1, 2] is to determine the largest positive integer $M(m, n)$ for which there is an m by n matrix of zeros and ones with $M(m, n)$ ones and no 2 by 2 submatrix consisting entirely of ones. Suppose we modify this problem by asking instead for the largest positive integer $N(m, n)$ for which there is an m by n matrix of zeros and ones with $N(m, n)$ ones and no "hexagon" consisting entirely of ones, rather than no "square". By this we mean that given any three rows r_1, r_2, r_3 and any three columns c_1, c_2, c_3 of a matrix, not all of the entries (r_1, c_1), (r_1, c_2), (r_2, c_2), (r_2, c_3), (r_3, c_1), and (r_3, c_3) are ones. Now we identify the columns of the matrix with the colors $c_1, \ldots, c_n$; the rows with the vertices $v_1, \ldots, v_m$ of the complete graph K_m and we interpret a pair of ones in the i^{th} column as indicating that the two associated vertices form an edge of color c_i. We see that the i^{th} column represents a complete subgraph of K_m colored with color c_i. The "hexagon" described above is nothing more than a cycle of length $r = 3$ whose edges are different colors. If N_i is the number of ones in the i^{th} column, then $\binom{N_i}{2}$ is the number of edges of color c_i. In searching for an upper bound for $N(m, n) = \sum N_i$, we may naturally look for an upper bound for $\sum \binom{N_i}{2}$, which is the number of incidences of edges of K_m and colors. Therefore, we address the following problem suggested by this interpretation.

Let K_m be the complete graph on m vertices, and let $c_1, \ldots, c_n$ be n distinct labels or colors. For each $i = 1, \ldots, n$ color some non-trivial complete subgraph of K_m with color c_i. We permit an edge of K_m to be colored with more than one color. Such a coloring will be called a multi-coloring of K_m. A multi-coloring will be called

r-admissible if there is no cycle of length r each edge of which has a different color. We denote by $E(m, n, r)$ the maximum, taken over all r-admissible colorings, of the number of incidences of edges of K_m and colors. We will show that

$$2\binom{m-1}{2} + n - 1 \leqslant E(m, n, 3) < 2\binom{m}{2} + n,$$

$$3\binom{m-2}{2} + 3n - 6 \leqslant E(m, n, 4) < 3\binom{m}{2} + 3n,$$

$$4\binom{m-3}{2} + 6n - 18 \leqslant E(m, n, 5) < 4\binom{m}{2} + 6n.$$

The above lower bounds are special cases of a more general inequality, namely

$$(r-1)\binom{m-r+2}{2} - \frac{(r-1)(r-2)^2}{2} + \binom{r-1}{2} n \leqslant E(m, n, r),$$

which we will also establish. If $n < r$ or $m < r$, then clearly any coloring is r-admissible and so $E(m, n, r) = n\binom{m}{2}$. In the sequel, we assume $n, m \geqslant r$.

The implication of these results to the aforementioned problem concerning "hexagons" is that:

$$\sum \binom{N_i}{2} \equiv E(m, n, 3) < 2\binom{m}{2} + n.$$

For any integer $p \geqslant 1$, this leads (by rearranging the inequality $0 \leqslant (1/2p)(N_i - p - 1)(N_i - p)$ and summing on i) to

$$N(m, n) = \sum N_i < \frac{1}{p} 2\binom{m}{2} + \left(\frac{1}{p} + \frac{p+1}{2}\right) n.$$

It is possible to obtain similar "matrix" interpretations for other values of r.

In the next sections, we establish the upper bound for $r = 3, 4$ and 5. Then we establish the lower bound for arbitrary r, and finally we mention some related problems.

2. The case $r = 3$

Theorem 2.1. $E(m, n, 3) < 2\binom{m}{2} + n$.

Proof: Suppose we are given a 3-admissible multi-coloring of K_m with $E(m, n, 3)$ number of incidences of edges and colors. For each $i = 1, \ldots, n$ let F_i be the subgraph of K_m whose edges are those edges of K_m that are colored with c_i and at least two other colors. Let the number of edges in F_i be f_i. By the definition of multi-coloring, every edge of K_m whose endpoints are in the vertex set of F_i has color c_i. Therefore, 3-admissibility implies that F_i can contain no path of length two.

Moreover, 3-admissibility implies that any edge of K_m not in F_i, but whose endpoints are in the vertex set of F_i, can be colored with no other color except c_i. It follows that if S is the number of edges of K_m with exactly one color, we have

$$\sum \binom{f_i}{2} \leq \tfrac{1}{4} S$$

and so

$$\sum f_i \leq \tfrac{1}{4} S + n.$$

But Σf_i is the number of incidences of colors and edges of K_m which have at least three colors, so if U is the number of edges of K_m with exactly two colors, we have

$$\begin{aligned} E(m, n, 3) &\leq \sum f_i + S + 2U \leq 2(S + U) + n \\ &\leq 2\binom{m}{2} + n \end{aligned}$$

Notice, however, that the last inequality will be strict unless $S + U = \binom{m}{2}$, in which case $f_i = 0$, for all i and so

$$E(m, n, 3) \leq S + 2U < 2\binom{m}{2} + n.$$

3. The case $r = 4$

Theorem 3.1. $E(m, n, 4) < 3\binom{m}{2} + 3n$

Proof: We proceed as in the proof of Theorem 2.1. Given a 4-admissible multi-coloring of K_m with $E(m, n, 4)$ number of incidences of colors and edges, we define F_i to be the subgraph of K_m with each edge having at least four colors, including c_i. As before, f_i is the number of edges of F_i. We will show that unless $f_i \leq 3$, there are at least $\frac{1}{2} f_i$ edges of K_m which are monochromatic of color c_i.

Here F_i can have no open path of length three, and so the non-trivial connected components of F_i are either stars (including 1-stars or edges) or triangles. Furthermore, every edge of K_m, not in F_i, but whose endpoints are in F_i, with the exception of those edges completing a triangle with a 2-star in F_i, must be monochromatic of color c_i. Therefore, if F_i contains t_i stars, p_i 2-stars, and hence $f_i + t_i$ vertices, we must have at least $\binom{f_i + t_i}{2} - f_i - p_i$ edges which are monochromatic of color c_i. For $f_i \geq 3$, this is at least $\frac{1}{2} f_i$. So letting S_i be the number of edges monochromatic of color c_i, we have $f_i \leq 2S_i + 3$. This gives us, with S being the number of monochromatic edges in K_m,

$$\sum f_i \leq 2S + 3n.$$

Then if B and T are the number of bichromatic and trichromatic edges in K_m, we get

$$E(m,n,4) \leqslant \Sigma f_i + S + 2B + 3T$$
$$\leqslant 3S + 2B + 3T + 3n$$
$$\leqslant 3\binom{m}{2} + 3n.$$

As before, strict inequality holds in the last inequality unless $f_i = 0$ for all i, in which case

$$E(m,n,4) \leqslant S + 2D + 3T < 3\binom{m}{2} + 3n.$$

4. The case $r = 5$

Theorem 4.1. $E(m,n,5) < 4\binom{m}{2} + 6n$

Proof: We proceed in a manner similar to the proofs of the previous theorems. Given a 5-admissible multi-coloring of K_m with $E(m,n,5)$ number of incidences of colors and edges, we define F_i to be the subgraph of K_m consisting of those edges having at least five colors, including c_i. Let f_i be the number of edges in F_i. Also; let B_i be the number of edges of K_m with endpoints in F_i, colored with c_i and at most one other color. We will show that $B_i \geqslant f_i - 6$.

We first notice that F_i can have no open path of length four. This means that the non-trivial connected components of F_i are of the following five types: (i) a k-star for $k \geqslant 1$, (ii) two stars whose centers are connected by an additional edge, (iii) a triangle, (iv) a triangle with some additional edges all connected to the same vertex of the triangle, and (v) a subgraph of K_4. Moreover, there is no open path of length four in K_m such that all of its vertices are in F_i, two or more of its edges are in F_i, and the remaining edges have at least three colors.

Consider first the case that all the components of F_i are 1-stars (edges). By the above remarks, it is not possible for a 1-star in F_i to be connected to more than one other 1-star in F_i by edges with at least three colors. Therefore

$$B_i \geqslant 4\binom{f_i}{2} - 4\left[\frac{f_i}{2}\right] \geqslant f_i - 6.$$

For the case that not all of the components of F_i are 1-stars, we proceed by induction on the number of components, say q. If $q = 1$, noticing that the result is trivial when $f_i \leqslant 6$, it is routine to check all possible configurations for F_i and verify $B_i \leqslant f_i - 6$. For when F_i is a k-star, $k \geqslant 7$, all edges in $K_m - F_i$ whose endpoints are in F_i, with at most one exception, can have at most one color excluding c_i. When F_i is a triangle with additional edges or two stars whose centers are connected no edge in $K_m - F_i$ whose endpoints are in F_i can have more than one color, excluding c_i. These are the only possibilities for F_i whenever $f_i \geqslant 7$. Assume the result is true for any F_i with less than q components. If F_i has $q \geq 2$ components, let G be a component with the minimum number of vertices, say $V(G)$. By the induction hypothesis, there are at least $f_i - E(G) - 6$ edges of K_m, whose endpoints lie in $F_i - G$, and which have color c_i and at most one other color. Now there is some component H of $F_i - G$ which contains an open path of length two. Therefore every edge of K_m connecting a vertex of G with an endpoint of this path is colored with c_i and at most one other color. There are $2V(G)$ such edges, and for any component of F_i, $2V(G) \geq E(G)$. Hence

$$B_i \geq f_i - E(G) - 6 + 2V(G) \geq f_i - 6.$$

To complete the proof of the theorem, we let B equal the number of edges of K_m colored with at most two colors, we let T be the number of edges of K_m with exactly three colors, and D be the number of edges with exactly four colors. Then

$$\begin{aligned} E(m,n,5) &\leq \sum f_i + 2B + 3T + 4D \\ &\leq 3B + 3T + 4D + 6n \\ &\leq 4\binom{m}{2} + 6n. \end{aligned}$$

The last inequality will be strict unless $f_i = 0$ for all i, in which case

$$E(m,n,5) \leq 2B + 3T + 4D < 4\binom{m}{2} + 6n.$$

5. The lower bounds

We will establish a lower bound for $E(m,n,r)$ by construction. Let K_m be the complete graph on the points $\{v_1, \ldots, v_m\}$. For $m \geq 2r - 2$ we proceed as follows. Color all edges of K_m with c_1. Color the complete graph on $\{v_1, \ldots, v_{m-r+1}\}$ with colors $c_2, \ldots, c_{r-1}$. Color the complete $r-1$ graph on $\{v_{m-r+2}, \ldots, v_m\}$ with the remaining colors $c_r, \ldots, c_n$. It is easy to see that there is no inadmissible cycle of length r. Therefore

$$\begin{aligned} E(m,n,r) &\geq \binom{m}{2} + (r-2)\binom{m-r+1}{2} + \binom{r-1}{2}(n-r+1) \\ &= (r-1)\binom{m-r+1}{2} - \frac{(r-2)^2(r-1)}{2} + \binom{r-1}{2}n. \end{aligned}$$

For $r \leq m \leq 2r-3$, color all edges of K_m with color c_1. Color the complete graph on $\{v_1, \ldots, v_{r-1}\}$ with all the remaining colors. Then

$$E(m,n,r) \geq \binom{m}{2} + (n-1)\binom{r-1}{2}.$$

The author has so far been unable to find any r-admissible multi-colorings which would imply that $E(m, n, r)$ is greater than this lower bound. However, if we remove the requirement that all the colors be used in the coloring we can improve, in some instances, on this lower bound. In particular, if we use exactly $r-1$ colors we can color each edge of K_m with each color, and not produce an inadmissible cycle of length r. Hence $E(m, n, r) \geq (r-1)\binom{m}{2}$ and this will be greater than the first lower bound for m sufficiently large compared to n.

This, by the way, indicates the reason for having the term $(r-1)\binom{m}{2}$ in the upper bounds of Theorems 2.1, 3.1 and 4.1. The proofs of these theorems do not require that all of the n colors be used in the multi-coloring. So we see from the above considerations that the right hand side of the inequalities must be at least $(r-1)\binom{m}{2}$.

6. Concluding remarks

Finally, we wish to mention some problems which are similar to the problem of this paper. One such problem arises if we change the definition of r-admissible to exclude the possibility of finding r points of K_m for which a different color can be chosen for each edge of the complete graph on these r points.

Another variation arises if we replace K_m by the complete bipartite graph $K(a, b)$ on two sets V_a and V_b and if we color some complete bipartite subgraph of $K(a, b)$ with color c_i for $i = 1, \ldots, n$. Then we may define an r, s-admissible multi-coloring to be one in which there is no set of r points from V_a and s points of V_b for which we can choose a different color for each edge of $K(r, s)$. While this problem does seem similar to the problem of this paper, the solution seems to be of a different nature. For instance, consider the complete bipartite graph $K(a, b)$ with $a = 2$. Choosing v in V_a, we color every edge of the complete bipartite subgraph $K(1, b)$ on $\{v\}$ and V_b with each color c_i for $i = 1, \ldots, n$. We therefore have a $2, k$-admissible multi-coloring for any k satisfying $2 \leq k \leq b$. Furthermore, the number of incidences of colors and edges of $K(2, b)$ is nb, which is one half of the maximum possible number of such incidences in any multi-coloring. So we see that there are multi-colorings of $K(a, b)$ with the property that the number of incidences is a constant times the product of the number of colors and the number of edges of $K(a, b)$. This will produce results of a rather different nature than the ones in Theorems 2.1, 3.1 and 4.1. It is therefore not surprising that the methods used in the proofs in these theorems do not seem to be effective for this problem.

At the present time, the author knows of no effective method which will deal with either of these problems.

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