

More on the Umbral Calculus, with Emphasis on the q -Umbral Calculus

STEVEN ROMAN

Department of Mathematics, California State University, Fullerton, California 92634

Submitted by G.-C. Rota

The interrelationship between distinct umbral calculi is studied. These ideas are applied in particular to q -umbral calculus, which shows how Andrews' q -theory relates to the q -theory discussed in the previous paper by the author. The q -Hermite polynomials and basic hypergeometric series are briefly discussed. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this paper we continue the study of the umbral calculus begun in [8] (see also [9, 10]). Some knowledge of Sections 1-5 and 11 of [8] would be helpful here.

Recall that for each sequence c_n of non-zero constants we defined a distinct umbral calculus, used in the study of polynomial sequences $s_n(x)$ whose generating function has the form

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} t^k = \frac{1}{g(\tilde{f}(t))} \varepsilon_x(\tilde{f}(t))$$

where $g(t) = g_0 + g_1 t + \dots$ ($g_0 \neq 0$), $\tilde{f}(t) = f_1 t + f_2 t^2 + \dots$ ($f_1 \neq 0$) and

$$\varepsilon_x(t) = \sum_{k=0}^{\infty} \frac{x^k}{c_k} t^k.$$

We called these sequences $s_n(x)$ Sheffer sequences. In Section 11 of [8] we touched briefly on the q -umbral calculus, defined by

$$c_n = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

Our objective in this paper is to begin studying the relationship between

distinct umbral calculi whose defining sequences c_n are related. We have restricted attention to the case $\tilde{f}(t) = t$.

The first step, in Section 2, is to extend slightly the notion of Sheffer sequence (for $\tilde{f}(t) = t$) to include sesquences $r_n(x)$ for which

$$\sum_{k=0}^{\infty} \frac{r_k(x)}{d_k} t^k = \frac{1}{g(t)} \varepsilon_x(t) \tag{1.1}$$

where $g(t)$ and $\varepsilon_x(t)$ are as above. Notice that the sequence d_n on the left may be different from the sequence c_n used in defining $\varepsilon_x(t)$. While this extension is modest from one point of view, namely that $(d_n/c_n)r_n(x)$ is Sheffer in the c_n -umbral calculus, it does serve a useful purpose. For example, the q -umbral calculus in [8] employs the identity

$$\varepsilon_y(t) s_n(x) = \sum_{k=0}^n \binom{n}{k}_q s_k(x) s_{n-k}(y)$$

whereas Andrews' q -theory [3] employs the identity

$$s_n(xy) = \sum_{k=0}^n \binom{n}{k}_q s_k(x) y^k s_{n-k}(xy).$$

Clarification of the roles of these identities is made possible by the aforementioned extension (see Theorem 2.5). We call sequences $r_n(x)$ satisfying (1.1) basic sequences, although this term is a bit overused.

In Section 3 we consider a class of operators whose importance in the umbral calculus was recognized by Andrews. In Section 4 we study the relationship between the exponential-type series $\varepsilon_x(t)$ for different umbral calculi. It is this series which lies at the heart of the theory.

Sections 5–7 are devoted to the q -umbral calculus and related umbral calculi. The q -Hermite polynomials, among others, are placed in the context of the umbral calculus. Infinite products are discussed briefly. Heine's theorem [11, p. 92] is shown to be nothing but a generating function and the basic analog of Gauss' theorem [11, p. 97] is shown to be but a special case of a result from Section 4. Space limitations force us to postpone to a sequel a more detailed discussion of these umbral calculi or of others.

In Section 8 we discuss very briefly some additional expansion formulas, including Carlitz's q -analog of a special case of the Lagrange inversion formula.

We wish to express our indebtedness to Andrews' work [3]. The influence of his ideas is clearly visible in this paper.

2. BASIC RESULTS

A sequence c_n of constants is *admissible* if $c_n \neq 0$ for all $n \geq 0$. We shall use boldface type to denote admissible sequences— \mathbf{c} for c_n , \mathbf{d} for d_n , etc.

For each admissible sequence \mathbf{c} we define linear functionals and operators, as in [8], by

$$\begin{aligned} \langle t_{\mathbf{c}}^k | x^n \rangle &= c_n \delta_{n,k}, \\ t_{\mathbf{c}}^k x^n &= \frac{c_n}{c_{n-k}} x^{n-k}, \quad k \leq n, \\ &= 0, \quad n < k. \end{aligned}$$

The subscript \mathbf{c} is needed since we shall be dealing with more than one admissible sequence at a time.

We recall from [8] that the sequence $s_n(x)$ is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$, where $g(t)$ is invertible, if and only if any one of the following equivalent conditions holds:

- (1) $\langle g(t_{\mathbf{c}}) t_{\mathbf{c}}^k | s_n(x) \rangle = c_n \delta_{n,k}$,
- (2) $\langle g(t_{\mathbf{c}}) | s_n(x) \rangle = c_0 \delta_{n,0}$, $t_{\mathbf{c}} s_n(x) = (c_n/c_{n-1}) s_{n-1}(x)$,
- (3) $\sum_{k=0}^{\infty} (s_k(x)/c_k) t^k = \varepsilon_{x,\mathbf{c}}(t)/g(t)$, where $\varepsilon_{x,\mathbf{c}}(t) = \sum_{k=0}^{\infty} (x^k/c_k) t^k$.

Notice that (3) is a formal equation in the formal variables x and t and so there is no need for a subscript on t . We say that $s_n(x)$ is *Sheffer for $t_{\mathbf{c}}$* if it is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$ for some invertible $g(t)$.

Let \mathbf{c} and \mathbf{d} be admissible sequences and let L be a linear functional for which $\langle L | 1 \rangle \neq 0$. We say that the sequence $p_n(x)$ is the *basic sequence* for $(t_{\mathbf{c}}, \mathbf{d}, L)$ if

- (1) $\deg p_n(x) = n$,
- (2) $\langle L | p_n(x) \rangle = d_0 \delta_{n,0}$,
- (3) $t_{\mathbf{c}} p_n(x) = (d_n/d_{n-1}) p_{n-1}(x)$.

For each choice of sequence c the linear functional L has a series representation $g_{L,\mathbf{c}}(t_{\mathbf{c}}) = \sum_{k=0}^{\infty} [\langle L | x^k \rangle / c_k] t_{\mathbf{c}}^k$. Thus as linear functionals $L = g_{L,\mathbf{c}}(t_{\mathbf{c}})$. Note that $g_{L,\mathbf{c}}(t)$ is invertible since $\langle L | 1 \rangle \neq 0$.

The following theorem is immediate from the definitions.

THEOREM 2.1. *The following are equivalent.*

- (i) $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$,
- (ii) $s_n(x) = (c_n/d_n) p_n(x)$ is Sheffer for $(g_{L,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$,
- (iii) $\langle g_{L,\mathbf{c}}(t_{\mathbf{c}}) t_{\mathbf{c}}^k | p_n(x) \rangle = d_n \delta_{n,k}$,

- (iv) $p_n(x) = (d_n/c_n) g_{L,c}(t_c)^{-1} x^n,$
- (v) $\sum_{k=0}^{\infty} (p_k(x)/d_k) t^k = \varepsilon_{y,c}(t)/g_{L,c}(t).$

Part (iii) of Theorem 2.1 gives us a version of the Expansion Theorem and its important corollary.

THEOREM 2.2. *If $p_n(x)$ is basic for (t_c, \mathbf{d}, L) then*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t_c) | p_k(x) \rangle}{d_k} g_{L,c}(t) t^k.$$

THEOREM 2.3. *If $p_n(x)$ is basic for (t_c, \mathbf{d}, L) then*

$$p(x) = \sum_{k \geq 0} \frac{\langle g_{L,c}(t_c) t_c^k | p(x) \rangle}{d_k} p_k(x).$$

A useful way to organize basic sequences is as follows. For each linear functional L satisfying $\langle L | 1 \rangle \neq 0$ we define the L -table or L -matrix to be the matrix whose rows and columns are indexed by the $2^{\mathbf{N}_0}$ possible admissible sequences and whose (\mathbf{c}, \mathbf{d}) entry is the basic sequence for (t_c, \mathbf{d}, L) , which we may denote by $B_{\mathbf{c},\mathbf{d}}(L)$. In L -table language, part (i) of Theorem 2.1 says that those entries of an L -table which are Sheffer sequences are precisely the diagonal entries $B_{\mathbf{c},\mathbf{c}}(L)$. According to part (ii) of this theorem, entries in the same row of an L -table are related in a simple manner. Borrowing notation from the old style umbral calculus we may write

$$B_{\mathbf{c},\mathbf{e}}(L) = \frac{\mathbf{e}}{\mathbf{d}} B_{\mathbf{c},\mathbf{d}}(L).$$

More exactly, if $p_n(x)$ is basic for (t_c, \mathbf{e}, L) and $r_n(x)$ is basic for (t_c, \mathbf{d}, L) then

$$p_n(x) = \frac{e_n}{d_n} r_n(x).$$

The columns of an L -table present more of a challenge than the rows. It is our intention to give an algebraic identity which characterizes these columns. We begin with a result of Andrews [3], in a slightly different form, which characterizes the operators t_c . Let σ_y be the linear operator on polynomials defined by

$$\sigma_y p(x) = p(xy).$$

THEOREM 2.4. *Let τ be a linear operator on polynomials. Then $\tau = t_c$ for some admissible sequence \mathbf{c} if and only if*

$$\tau x^n \neq 0 \quad \text{all } n > 0$$

$$\tau \sigma_y = y \sigma_y \tau$$

for all constants y .

Proof. One easily sees that $t_c \sigma_y x^n = y \sigma_y t_c x^n$ and $t_c x^n \neq 0$ for $n > 0$. For the converse, set $\tau x^n = s_n(x)$. Then $y s_n(xy) = y \sigma_y s_n(x) = y \sigma_y \tau x^n = \tau \sigma_y x^n = y^n \tau x^n = y^n s_n(x)$. Setting $x = 1$ and changing y to x gives

$$\tau x^n = s_n(x) = s_n(1) x^{n-1}, \quad n > 0,$$

$$\tau 1 = s_0(x) = 0.$$

Since $s_n(1) \neq 0$ (otherwise $\tau x^n = 0$) we may set

$$c_n = s_1(1) s_2(1) \cdots s_n(1) c_0$$

with $c_0 \neq 0$ arbitrary. Then $\tau x^n = (c_n/c_{n-1}) x^{n-1} = t_c x^n$. Thus $\tau = t_c$.

We are now ready to characterize the columns of an L -table.

THEOREM 2.5. *The sequence $p_n(x)$ is basic for (t_c, \mathbf{d}, L) for some admissible sequence \mathbf{c} if and only if $\deg p_n(x) = n$ and*

$$p_n(xy) = \sum_{k=0}^n \frac{d_n}{d_k d_{n-k}} p_k(x) y^k \langle L | p_{n-k}(xy) \rangle \tag{2.1}$$

for all constants y . In other words, $p_n(x)$ is in the \mathbf{d} th column of the L -table if and only if it satisfies (2.1).

Proof. First assume $p_n(x)$ is basic for (t_c, \mathbf{d}, L) . The Expansion Theorem gives

$$p_n(xy) = \sum_{k=0}^n \frac{p_k(xy)}{c_k} \langle g_{L,c}(t_c) t_c^k | p_n(xy) \rangle.$$

But

$$\begin{aligned} \langle g_{L,c}(t_c) t_c^k | p_n(xy) \rangle &= \langle g_{L,c}(t_c) | t_c^k \sigma_y p_n(x) \rangle \\ &= y^k \langle g_{L,c}(t_c) | \sigma_y t_c^k p_n(x) \rangle \\ &= \frac{d_n}{d_{n-k}} y^k \langle L | \sigma_y p_{n-k}(x) \rangle \end{aligned}$$

and the result follows. For the converse, suppose $\deg p_n(x) = n$ and $p_n(x)$

satisfies (2.1). Then let τ be the linear operator defined by $\tau p_n(x) = (d_n/d_{n-1}) p_{n-1}(x)$. Now (2.1) can be written as

$$\sigma_y p_n(x) = \sum_{k=0}^n \frac{d_n}{d_k d_{n-k}} p_{n-k}(x) y^{n-k} \langle L | p_k(xy) \rangle$$

and applying τ gives

$$\begin{aligned} \tau \sigma_y p_n(x) &= \sum_{k=0}^{n-1} \frac{d_n}{d_k d_{n-k}} \frac{d_{n-k}}{d_{n-k-1}} p_{n-k-1}(x) y^{n-k} \langle L | p_k(xy) \rangle \\ &= y \frac{d_n}{d_{n-1}} \sum_{k=0}^{n-1} \frac{d_{n-1}}{d_k d_{n-1-k}} p_{n-1-k}(x) y^{n-1-k} \langle L | p_k(xy) \rangle \\ &= y \frac{d_n}{d_{n-1}} \sigma_y p_{n-1}(x) \\ &= y \sigma_y \tau p_n(x). \end{aligned}$$

Thus $\tau \sigma_y = y \sigma_y \tau$ and since $\tau x^n \neq 0$ we deduce from Theorem 2.4 the existence of an admissible sequence \mathbf{c} for which $\tau = t_{\mathbf{c}}$. Thus $t_{\mathbf{c}} p_n(x) = (d_n/d_{n-1}) p_{n-1}(x)$. Finally, setting $y = 1$ in (2.1) and comparing coefficients of $p_k(x)$ on both sides gives $\langle L | p_n(x) \rangle = d_0 \delta_{n,0}$. Hence $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$ and the proof is complete.

Let us consider Andrews' q -theory in the present context. Comparing his Definition 1 in [3, p. 349] with (2.1) one sees that his theory is devoted to the d th column of the ε_1 table where ε_1 is evaluation at $x=1$ and $d_n = (1-q) \cdots (1-q^n)/(1-q)^n$. Incidentally, this explains why there is no transfer formula [8, Eq. (7.1)] in Andrews' q -theory, for this formula reduces to triviality when $\tilde{f}(t) = t$. Further, there is a recurrence formula for basic sequences, obtained directly from [8, Eq. (6.3)] but unfortunately it seems difficult to employ usefully in the present context.

3. INVARIANT OPERATORS

Let us call a linear operator τ *invariant* if

$$\tau \sigma_y = \sigma_y \tau$$

for all constants y . Andrews [3] used the term Eulerian shift-invariant. Some examples of invariant operators are:

- (1) σ_y and any linear combination of such operators,
- (2) $x t_{\mathbf{c}}$ and any polynomial in $x t_{\mathbf{c}}$,
- (3) $x^k t_{\mathbf{c}}^k$.

With regard to the last example, we state without proof (which follows the lines of that for Theorem 2.4) the following result.

THEOREM 3.1. *Let τ be invariant and suppose $\tau x^n \neq 0$ for $k \leq n$ but $\tau x^n = 0$ for $n < k$. Then $\tau = x^k t_c^k$ for some c .*

Invariant operators have their own Expansion Theorem. If L is a linear functional then by L_y we mean the linear operator acting on polynomials in x and y defined by

$$L_y x^n y^m = \langle L | x^m \rangle x^n.$$

We also define the operator \bar{L} , acting on polynomials in x , by

$$\bar{L}x^n = \langle L | x^n \rangle x^n.$$

Notice that $\bar{L}x^n = L_y x^n y^n = L_y \sigma_y x^n$ and so $\bar{L} = L_y \sigma_y$.

THEOREM 3.2. *Let τ be invariant and suppose that $p_n(x)$ is basic for (t_c, \mathbf{d}, L) . Then*

$$\tau = \sum_{k=0}^{\infty} \frac{1}{d_k} \langle \varepsilon_1 | \tau p_k(x) \rangle x^k \bar{L}t_c^k$$

where ε_1 is evaluation at $x = 1$.

Proof. We first observe that $\tau \sigma_y p_n(x) = \sigma_y \tau p_n(x)$ is symmetric in x and y and so, treating y as a variable, we have $\tau \sigma_y p_n(x) = \tau_y (\sigma_x)_y p_n(y) = \tau_y p_n(xy)$. Using Eq. (2.1) with x and y interchanged we have

$$\begin{aligned} \tau p_n(x) &= \langle (\varepsilon_1)_y | \sigma_y \tau p_n(x) \rangle \\ &= \langle (\varepsilon_1)_y | \tau_y p_n(xy) \rangle \\ &= \sum_{k=0}^n \frac{d_n}{d_k d_{n-k}} \langle (\varepsilon_1)_y | \tau_y p_k(y) \rangle x^k L_y p_{n-k}(xy) \\ &= \sum_{k=0}^n \frac{d_n}{d_k d_{n-k}} \langle \varepsilon_1 | \tau p_k(x) \rangle x^k L_y \sigma_y p_{n-k}(x) \\ &= \sum_{k=0}^n \frac{1}{d_k} \langle \varepsilon_1 | \tau p_k(x) \rangle x^k L_y \sigma_y t_c^k p_n(x). \end{aligned}$$

The result follows.

Since \bar{L} is the identity when $L = \varepsilon_1$ we obtain the following corollaries.

COROLLARY 3.3. *Let τ be invariant and suppose $p_n(x)$ is basic for $(t_c, \mathbf{d}, \varepsilon_1)$. Then*

$$\tau = \sum_{k=0}^{\infty} \frac{1}{d_k} \langle \varepsilon_1 | \tau p_k(x) \rangle x^k t_c^k.$$

COROLLARY 3.4. *If $p_n(x)$ is basic for $(t_c, \mathbf{d}, \varepsilon_1)$ then*

$$\sigma_z = \sum_{k=0}^{\infty} \frac{1}{d_k} p_k(z) x^k t_c^k.$$

Before we leave this section we observe that

$$\begin{aligned} \langle \sigma_y^* t_c^k | x^n \rangle &= \langle t_c^k | \sigma_y x^n \rangle \\ &= y^n c_n \delta_{n,k} \\ &= \langle (yt_c)^k | x^n \rangle \end{aligned}$$

and so

$$\sigma_y^* f(t) = f(yt).$$

In other symbols.

$$\langle f(yt_c) | p(x) \rangle = \langle f(t_c) | p(xy) \rangle.$$

4. EXPONENTIAL-TYPE SERIES

In this section we shall consider the series

$$\varepsilon_{y,c}(t) = \sum_{k=0}^{\infty} \frac{y^k}{c_k} t^k.$$

This is the “exponential” series in the \mathbf{c} -umbral calculus. We define the linear operator ∂_c acting on power series in t by

$$\partial_c t^k = \frac{c_k}{c_{k-1}} t^{k-1}.$$

Thus, for example, $\langle \partial_c t_c^k | x^n \rangle = (c_k/c_{k-1}) \langle t_c^{k-1} | x^n \rangle = (c_k/c_{k-1}) d_{k-1} \delta_{k-1,n}$. It is easy to see [take $f(t) = t^k$ and $p(x) = x^n$] that

$$\langle f(t_c) | xp(x) \rangle = \langle \partial_c f(t_c) | p(x) \rangle.$$

Also

$$\partial_c \varepsilon_{y,c}(t) = y \varepsilon_{y,c}(t).$$

Our next goal is to get some information about $\partial_{\mathbf{d}}\varepsilon_{y,c}(t)$. One easily verifies the next lemma by setting $p(x) = x^n$.

LEMMA 4.1. $\langle t_c \partial_{\mathbf{d}} \varepsilon_{y,c}(t_c) \mid p(x) \rangle = \langle \varepsilon_1 \mid t_{\mathbf{d}} p(xy) \rangle$.

THEOREM 4.2. *Let $s_n(x)$ be Sheffer for $(\varepsilon_{y,c}(t_c), t_c)$. Then*

$$\partial_{\mathbf{d}} \varepsilon_{y,c}(t) = \left(\sum_{k=0}^{\infty} \frac{1}{c_{k+1}} \langle \varepsilon_1 \mid t_{\mathbf{d}} s_{k+1}(xy) \rangle t^k \right) \varepsilon_{y,c}(t).$$

Proof. By the Expansion Theorem

$$\begin{aligned} t \partial_{\mathbf{d}} \varepsilon_{y,c}(t) &= \sum_{k=1}^{\infty} \frac{1}{c_k} \langle t_c \partial_{\mathbf{d}} \varepsilon_{y,c}(t_c) \mid s_k(x) \rangle \varepsilon_{y,c}(t) t^k \\ &= \sum_{k=1}^{\infty} \frac{1}{c_k} \langle \varepsilon_1 \mid t_{\mathbf{d}} s_k(xy) \rangle \varepsilon_{y,c}(t) t^k \end{aligned}$$

from which the result follows.

If we take $d_n = n!$ then $t_{\mathbf{d}}$ is the ordinary derivative D and Theorem 4.2 gives

$$D\varepsilon_{y,c}(t) = \left(\sum_{k=0}^{\infty} \frac{1}{c_{k+1}} \langle \varepsilon_1 \mid Ds_{k+1}(xy) \rangle t^k \right) \varepsilon_{y,c}(t).$$

Solving this differential equation we have

THEOREM 4.3. *Let $s_n(x)$ be Sheffer for $(\varepsilon_{y,c}(t_c), t_c)$. Then*

$$\varepsilon_{y,c}(t) = \exp \sum_{k=1}^{\infty} \frac{1}{kc_k} \langle \varepsilon_1 \mid Ds_k(xy) \rangle t^k.$$

A situation which seems to occur frequently is when two admissible sequences are related by

$$d_n = \frac{c_n}{\langle N \mid u_n(x) \rangle}$$

where $u_n(x)$ is a sequence of polynomials and N is a linear functional for which $\langle N \mid u_n(x) \rangle \neq 0$. In this case

$$\begin{aligned} \varepsilon_{y,d}(t) &= \sum_{k=0}^{\infty} \frac{y^k}{d_k} t^k \\ &= \sum_{k=0}^{\infty} \frac{\langle N \mid u_k(x) \rangle}{c_k} (yt)^k \\ &= \left\langle N \mid \sum_{k=0}^{\infty} \frac{u_k(x)}{c_k} (yt)^k \right\rangle \end{aligned}$$

where the last equality is, in effect, a definition. This leads to the following result.

THEOREM 4.4. *Let \mathbf{c} be an admissible sequence, $u_n(x)$ the basic sequence for $(t_{\mathbf{c}}, \mathbf{c}, M)$ and N a linear functional for which $\langle N | u_n(x) \rangle \neq 0$. Then if*

$$d_n = \frac{c_n}{\langle N | u_n(x) \rangle}$$

we have

$$\varepsilon_{y,d}(t) = \frac{\langle N | \varepsilon_{x,c}(yt) \rangle}{g_{M,c}(yt)}.$$

Next we consider the problem of determining $D_k(t)$ in the formal equation

$$\varepsilon_{x,d}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} D_k(t)$$

where $s_n(x)$ is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$. Clearly $D_k(t)$ exists and in fact

$$D_k(t) = \langle g(t_{\mathbf{c}}) t_{\mathbf{c}}^k | \varepsilon_{x,d}(t) \rangle$$

where t is a formal variable and the linear functional $g(t_{\mathbf{c}}) t_{\mathbf{c}}^k$ acts on the coefficients of the powers of t (which are polynomials in x).

Let us suppose that the operators $t_{\mathbf{c}}$ and $t_{\mathbf{d}}$ are related by

$$t_{\mathbf{c}} = u(\sigma_y) t_{\mathbf{d}}$$

where $u(\sigma_y)$ is a linear combination of integral powers of σ_y . Since $t_{\mathbf{d}} \sigma_y^n = y^n \sigma_y^n t_{\mathbf{d}}$ we have

$$t_{\mathbf{c}}^k = u(\sigma_y) u(y\sigma_y) \cdots u(y^{k-1}\sigma_y) t_{\mathbf{d}}^k.$$

Since $t_{\mathbf{d}}^k \varepsilon_{x,d}(t) = t^k \varepsilon_{x,d}(t)$ we get

$$\begin{aligned} D_k(t) &= \langle g(t_{\mathbf{c}}) | t_{\mathbf{c}}^k \varepsilon_{x,d}(t) \rangle \\ &= t^k \langle g(t_{\mathbf{c}}) | u(\sigma_y) \cdots u(y^{k-1}\sigma_y) \varepsilon_{x,d}(t) \rangle. \end{aligned}$$

Now if

$$l(t) = \sum_{k=0}^{\infty} \frac{\langle g(t_{\mathbf{c}}) | x^k \rangle}{d_k} t^k$$

then $\langle l(t_{\mathbf{d}}) \mid p(x) \rangle = \langle g(t_{\mathbf{c}}) \mid p(x) \rangle$ and so

$$\begin{aligned} D_k(t) &= t^k \langle l(t_{\mathbf{d}}) \mid u(\sigma_y) \cdots u(y^{k-1}\sigma_y) \varepsilon_{x,\mathbf{d}}(t) \rangle \\ &= t^k \langle u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*) l(t_{\mathbf{d}}) \mid \varepsilon_{x,\mathbf{d}}(t) \rangle. \end{aligned}$$

Finally, since for any series $f(t)$ we have $\langle f(t_{\mathbf{d}}) \mid \varepsilon_{x,\mathbf{d}}(t) \rangle = f(t)$ we get

$$D_k(t) = t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*) l(t).$$

We have proved the following result.

THEOREM 4.5. *Let \mathbf{c} and \mathbf{d} be admissible sequences such that*

$$t_{\mathbf{c}} = u(\sigma_y) t_{\mathbf{d}}$$

where $u(\sigma_y)$ is a finite linear combination of integral powers of σ_y . Then if $s_n(x)$ is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$ we have

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*) l(t)$$

where

$$l(t) = \sum_{k=0}^{\infty} \frac{\langle g(t_{\mathbf{c}}) \mid x^k \rangle}{d_k} t^k.$$

COROLLARY 4.6. *Let \mathbf{c} and \mathbf{d} be as in Theorem 4.5 and suppose $s_n(x)$ is Sheffer for $(\varepsilon_{z,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$. Then*

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*) \varepsilon_{z,\mathbf{d}}(t).$$

5. THE q -CASE

Let us first set down the basic facts of the q -case. We set

$$c_n = \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-q)^n}.$$

Then

$$\frac{c_n}{c_{n-1}} = \frac{1-q^n}{1-q}$$

and

$$\frac{c_n}{c_k c_{n-k}} = \binom{n}{k}_q$$

is the *q*-binomial coefficient.

The operator t_c is known as the *q*-derivative and we have

$$t_c x^n = \frac{1 - q^n}{1 - q} x^{n-1}$$

and so

$$t_c p(x) = \frac{p(x) - p(qx)}{(1 - q)x}.$$

Thus if $s_n(x)$ is Sheffer for t_c we immediately have the recurrence

$$s_n(x) - s_n(qx) = (1 - q^n) x s_{n-1}(x). \tag{5.1}$$

The operator t_c is related to σ_q by

$$\sigma_q = 1 - (1 - q) x t_c.$$

The operator ∂_c is given by

$$\partial_c f(t) = \frac{f(t) - f(qt)}{(1 - q)t} \tag{5.2}$$

and satisfies the Leibnitz formula (see [8] for an umbral proof)

$$\partial_c^n f(t) g(t) = \sum_{k=0}^n \binom{n}{k}_q q^{k(k-n)} \partial_c^k f(t) \partial_c^{n-k} g(q^k t).$$

The *q*-exponential series

$$\varepsilon_{y,c}(t) = \sum_{k=0}^{\infty} \frac{1}{c_k} (yt)^k$$

satisfies $\partial_c \varepsilon_{y,c}(t) = y \varepsilon_{y,c}(t)$, which using (5.2) is

$$\varepsilon_{y,c}(qt) = (1 - (1 - q)yt) \varepsilon_{y,c}(t).$$

We also note that

$$\begin{aligned} \varepsilon_{y,c}(t) &= \varepsilon_{1,c}(yt) \\ \partial_c \varepsilon_{y,c}(t)^{-1} &= -y \varepsilon_{y,c}(qt)^{-1} \\ \partial_c \varepsilon_{y,c}(t) \varepsilon_{z,c}(t) &= (y + z - (1 - q)yz) \varepsilon_{y,c}(t) \varepsilon_{z,c}(t). \end{aligned} \tag{5.3}$$

The q -binomial coefficients have important combinatorial significance [6, 7]. In [7] it is shown that when q is a prime power the number $\binom{n}{k}_q$ is the number of vector subspaces of dimension k of a vector space of dimension n over the field $GF[q]$. Following [6] we set

$$G_n = \sum_{k=0}^n \binom{n}{k}_q,$$

$$A_n = \sum_{k=0}^n (-1)^k \binom{n}{k}_q.$$

Then G_n is the number of subspaces of an n -dimensional vector space over $GF[q]$.

Now we first observe that

$$\left\langle \varepsilon_{1,c}(t_c) \frac{t_c^k}{c_k} \middle| x^n \right\rangle = \binom{n}{k}_q.$$

From this we obtain some of the simplest properties of $\binom{n}{k}_q$,

$$\begin{aligned} \binom{n}{k}_q &= \left\langle \varepsilon_{1,c}(t_c) \frac{t_c^k}{c_k} \middle| x^n \right\rangle \\ &= \frac{c_{k-1}}{c_k} \left\langle \varepsilon_{1,c}(t_c) \frac{t_c^{k-1}}{c_{k-1}} \middle| t_c x^n \right\rangle \\ &= \frac{c_{k-1}}{c_k} \frac{c_n}{c_{n-1}} \binom{n-1}{k-1}_q \\ &= \frac{1-q^n}{1-q^k} \binom{n-1}{k-1}_q \end{aligned}$$

and

$$\begin{aligned} \binom{n}{k}_q &= \left\langle \varepsilon_{1,c}(t_c) \frac{t_c^k}{c_k} \middle| x^n \right\rangle \\ &= \left\langle \partial_c \varepsilon_{1,c}(t_c) \frac{t_c^k}{c_k} \middle| x^{n-1} \right\rangle \\ &= \left\langle \varepsilon_{1,c}(t_c) \frac{t_c^{k-1}}{c_{k-1}} + \frac{q^k t_c^k}{c_k} \varepsilon_{1,c}(t_c) \middle| x^{n-1} \right\rangle \\ &= \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q. \end{aligned}$$

Next, using (5.3) we have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k}_q y^k z^{n-k} &= \langle \varepsilon_{y,c}(t_c) \varepsilon_{z,c}(t_c) | x^n \rangle \\ &= \langle (y+z-(1-q)yz) \varepsilon_{y,c}(t_c) \varepsilon_{z,c}(t_c) | x^{n-1} \rangle \\ &= (y+z) \sum_{k=0}^{n-1} \binom{n-1}{k}_q y^k z^{n-1-k} \\ &\quad - (1-q^{n-1}) yz \sum_{k=0}^{n-2} \binom{n-2}{k}_q y^k z^{n-2-k}. \end{aligned}$$

If we take $y=z=1$ we get the recurrence

$$G_n = 2G_{n-1} - (1-q^{n-1})G_{n-2}$$

and if we take $y=-1$ and $z=1$ we get

$$A_n = (1-q^{n-1})A_{n-2}.$$

Basic umbral techniques can be used to determine solutions to certain recurrence relations. For example, in [2] Al-Salam determines those sequences $s_n(x)$, Sheffer for t_c , which are orthogonal, that is, which satisfy the recurrence

$$s_{n+1}(x) = (x - b_n) s_n(x) - d_n s_{n-1}(x), \tag{5.4}$$

$s_0(x)=1$. After determining b_n and d_n , Al-Salam simply states the generating function of the solution. Let us give a complete, yet simple, umbral solution. Suppose $s_n(x)$ satisfies (5.4) and is Sheffer for $(g(t_c), t_c)$. Applying the linear functional $g(t_c) t_c^k$ to (5.4) gives

$$\begin{aligned} \langle g(t_c) t_c^k | s_{n+1}(x) \rangle &= \langle g(t_c) t_c^k | (x - b_n) s_n(x) \rangle \\ &\quad - \langle g(t_c) t_c^k | d_n s_{n-1}(x) \rangle. \end{aligned} \tag{5.5}$$

Then we observe that

$$\begin{aligned} \langle g(t_c) t_c^k | s_{n+1}(x) \rangle &= c_k \delta_{k,n+1} = \left\langle \frac{c_k}{c_{k-1}} g(t_c) t_c^{k-1} | s_n(x) \right\rangle, \\ \langle g(t_c) t_c^k | x s_n(x) \rangle &= \langle \partial_c g(t_c) t_c^k | s_n(x) \rangle \end{aligned}$$

and

$$\langle g(t_c) t_c^k | s_{n-1}(x) \rangle = c_k \delta_{k,n-1} = \left\langle \frac{c_k}{c_{k+1}} g(t_c) t_c^{k+1} | s_n(x) \right\rangle.$$

Thus from (5.5) we get

$$\frac{c_k}{c_{k-1}} g(t) t^{k-1} = \partial_c g(t) t^k - b_k g(t) t^k - \frac{c_k}{c_{k+1}} d_{k+1} g(t) t^{k+1}.$$

Using Leibniz' formula $\partial_c g(t) t^k = (c_k/c_{k-1}) g(t) t^{k-1} + q^k t^k \partial_c g(t)$, and rearranging gives

$$\partial_c g(t) = q^{-k} \left(b_k + \frac{c_k}{c_{k+1}} d_{k+1} t \right) g(t),$$

this for all $k \geq 0$. Therefore

$$b_k = q^k b_0$$

$$d_{k+1} = \frac{c_{k+1}}{c_k} q^k d_1$$

and

$$\partial_c g(t) = (b_0 + d_1 t) g(t).$$

Recalling Eq. (5.3) we see that

$$g(t) = \varepsilon_{y,c}(t) \varepsilon_{z,c}(t)$$

where $y + z = b_0$ and $(1 - q)yz = d_1$. The generating function for $s_n(x)$ is thus $\varepsilon_{x,c}(t)/\varepsilon_{y,c}(t) \varepsilon_{z,c}(t)$.

In [2] Al-Salam also shows that those sequences Sheffer for t_c and orthogonal on the unit circle are characterized by satisfying the recurrence

$$s_{n+1}(x) = (x + \gamma + \beta q^n) s_n(x) - \gamma(1 - q^n) x s_{n-1}(x). \quad (5.6)$$

An umbral solution to this recurrence is as follows. Using (5.1) in (5.6) we get

$$s_{n+1}(x) = (x + \beta q^n) s_n(x) + \gamma s_n(qx).$$

If $s_n(x)$ is Sheffer for $(g(t_c), t_c)$ then applying the linear functional $g(t_c)$ gives

$$0 = \langle \partial_c g(t_c) | s_n(x) \rangle + \beta q^n \langle g(t_c) | s_n(x) \rangle + \gamma \langle g(qt_c) | s_n(x) \rangle.$$

Now $q^n \langle g(t_c) | s_n(x) \rangle = q^n c_n \delta_{n,0} = c_n \delta_{n,0} = \langle g(t_c) | s_n(x) \rangle$ and so, as before,

$$\partial_c g(t) = -\beta q(t) - \gamma g(qt).$$

Using (5.2) we get

$$\frac{g(qt)}{g(t)} = \frac{1 + (1 - q) \beta t}{1 + (1 - q) \gamma t}.$$

Finally, recalling that $\varepsilon_{y,c}(qt)/\varepsilon_{y,c}(t) = 1 - (1 - q) \gamma t$ we have

$$g(t) = \frac{\varepsilon_{-\beta,c}(t)}{\varepsilon_{-\gamma,c}(t)}$$

and the generating function for $s_n(x)$ is $\varepsilon_{-\gamma,c}(t) \varepsilon_{x,c}(t)/\varepsilon_{-\beta,c}(t)$.

Let us return to some examples of Sheffer sequences.

1. We write $s_n(x) = [x]_{y,n}$ for the Sheffer sequence for $(\varepsilon_{y,c}(t_c), t_c)$. To determine $[x]_{y,n}$ we first observe that $[y]_{y,n+1} = \langle \varepsilon_{y,c}(t_c) | [x]_{y,n+1} \rangle = 0$ and so we may set

$$[x]_{y,n+1} = (x - y) r_n(x).$$

Now Leibniz' rule gives

$$\begin{aligned} c_{k+1} \delta_{n+1,k+1} &= \langle \varepsilon_{y,c}(t_c) t_c^{k+1} | [x]_{y,n+1} \rangle \\ &= \langle \varepsilon_{y,c}(t_c) t_c^{k+1} | (x - y) r_n(x) \rangle \\ &= \langle (\partial_c - y) \varepsilon_{y,c}(t_c) t_c^{k+1} | r_n(x) \rangle \\ &= \frac{c_{k+1}}{c_k} \langle \varepsilon_{y,c}(qt_c) t_c^k | r_n(x) \rangle \end{aligned}$$

and since $\varepsilon_{y,c}(qt) = \varepsilon_{qy,c}(t)$ we get

$$\langle \varepsilon_{qy,c}(t_c) t_c^k | r_n(x) \rangle = c_k \delta_{n,k}.$$

Thus $r_n(x) = [x]_{qy,n}$ and so

$$[x]_{y,n+1} = (x - y)[x]_{qy,n}$$

leading to

$$[x]_{y,n} = (x - y)(x - qy) \cdots (x - q^{n-1}y).$$

The generating function for $[x]_{y,n}$ is

$$\sum_{k=0}^{\infty} \frac{[x]_{y,k}}{c_k} t^k = \frac{\varepsilon_{x,c}(t)}{\varepsilon_{y,c}(t)}.$$

Setting $x = 0$ and noticing that

$$[0]_{y,k} = (-y)^k q^{\binom{k}{2}}$$

we get

$$\frac{1}{\varepsilon_{y,c}(t)} = \sum_{k=0}^{\infty} \frac{(-y)^k}{c_k} q^{\binom{k}{2}} t^k. \tag{5.7}$$

Equation (5.1) is the recurrence

$$[x]_{y,n} - [qx]_{y,n} = (1 - q^n) x [x]_{y,n-1}.$$

Theorem 2.5 becomes

$$[xy]_{z,n} = \sum_{k=0}^n \binom{n}{k}_q [x]_{z,k} y^k [yz]_{z,n-k}.$$

The exponential $\varepsilon_{y,c}(t)$ is related to infinite products. With regard to Theorem 4.3, setting $y = 1$, we get

$$\begin{aligned} \langle \varepsilon_1 | D[x]_k \rangle &= \langle \varepsilon_1 | D(x-1) \cdots (x-q^{k-1}) \rangle \\ &= (1-q) \cdots (1-q^{k-1}) \\ &= \frac{(1-q)^k}{1-q^k} c_k \end{aligned}$$

and so, as Andrews [3] has observed,

$$\begin{aligned} \varepsilon_{1,c}(t) &= \exp \sum_{k=1}^{\infty} \frac{1}{1-q^k} \frac{[(1-q)t]^k}{k} \\ &= \exp \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} q^{jk} \frac{[(1-q)t]^k}{k} \\ &= \exp \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{[q^j(1-q)t]^k}{k} \\ &= \exp \sum_{j=0}^{\infty} \log(1 - (1-q)q^j t)^{-1} \\ &= \prod_{j=0}^{\infty} \frac{1}{1 - (1-q)q^j t}. \end{aligned}$$

Thus

$$\varepsilon_{1,c}(t/(1-q)) = \prod_{j=0}^{\infty} \frac{1}{1 - q^j t}.$$

The above calculations can be justified on various grounds, for example, if $|q| < 1$ and $|t| < 1/(1 - q)$.

Combining this with the generating function for $[x]_{y,n}$ we have

$$\prod_{j=0}^{\infty} \frac{1 - (1 - q) q^j y^j}{1 - (1 - q) q^j x^j} = \sum_{k=0}^{\infty} \frac{[x]_{y,k}}{c_k} t^k.$$

Replacing t by $t/(1 - q)$ and using the notation of basic hypergeometric series, $c_k(1 - q)^k = (q; q)_k$ and $[x]_{y,k} = x^k(y/x; q)_k$, this becomes Heine's theorem [11, p. 92]

$$\prod_{j=0}^{\infty} \frac{1 - q^j y^j}{1 - q^j z^j} = \sum_{k=0}^{\infty} \frac{(y/z; q)_k}{(q; q)_k} (xt)^k = {}_1\phi_0[y/x; q; xt].$$

The Expansion Theorem gives

$$\begin{aligned} [x]_{y,n} &= \sum_{k=0}^n \frac{\langle t_c^k | [x]_{y,n} \rangle}{c_k} x^k \\ &= \sum_{k=0}^n \binom{n}{k}_q \langle \varepsilon_{0,c}(t_c) | [x]_{y,n-k} \rangle x^k \\ &= \sum_{k=0}^n \binom{n}{k}_q (-y)^{n-k} q^{\binom{n-k}{2}} x^k \end{aligned}$$

and

$$\begin{aligned} x^n &= \sum_{k=0}^n \frac{\langle \varepsilon_{y,c}(t_c) t_c^k | x^n \rangle}{c_k} [x]_{y,k} \\ &= \sum_{k=0}^n \binom{n}{k}_q y^{n-k} [x]_{y,k}. \end{aligned}$$

The Expansion Theorem also gives

$$x[x]_{y,n} = \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,c}(t_c) t_c^k | x[x]_{y,n} \rangle}{c_k} [x]_{y,k}.$$

But

$$\begin{aligned} \langle \varepsilon_{y,c}(t_c) t_c^k | x[x]_{y,n} \rangle &= \langle \partial_c \varepsilon_{y,c}(t_c) t_c^k | [x]_{y,n} \rangle \\ &= \left\langle \frac{c_k}{c_{k-1}} \varepsilon_{y,c}(t_c) t_c^{k-1} \right. \\ &\quad \left. + (qt_c)^k y \varepsilon_{y,c}(t_c) | [x]_{y,n} \right\rangle \\ &= c_k \delta_{n,k-1} + q^k y c_k \delta_{k,n} \end{aligned}$$

and so

$$x[x]_{y,n} = [x]_{y,n+1} + q^n y [x]_{y,n}.$$

In the Expansion Theorem for invariant operators (Theorem 3.2) let us put $p_n(x) = [x]_{1,n} = [x]_n$. Then $L = \varepsilon_1$ and

$$\tau = \sum_{k=0}^{\infty} \frac{1}{c_k} \langle \varepsilon_1 | \tau [x]_k \rangle x^k t_c^k.$$

In particular, if $\tau = \sigma_q^m$ then for $k \geq m + 1$ we have $[q^m]_k = 0$ and for $k \leq m$ we have

$$\begin{aligned} [q^m]_k &= (q^m - 1) \cdots (q^m - q^{k-1}) \\ &= (q - 1)^k q^{\binom{k}{2}} \frac{c_m}{c_{m-k}} \\ &= \langle (q - 1)^k q^{\binom{k}{2}} \varepsilon_{1,c}(t_c) t_c^k | x^m \rangle. \end{aligned}$$

Thus

$$\sigma_q^m = \sum_{k=0}^m \frac{(q - 1)^k}{c_k} q^{\binom{k}{2}} \langle \varepsilon_{1,c}(t_c) t_c^k | x^m \rangle x^k t_c^k.$$

For any polynomial $p(x)$ we get

$$p(\sigma_q) = \sum_{k=0}^{\infty} \frac{(q - 1)^k}{c_k} q^{\binom{k}{2}} \langle \varepsilon_{1,c}(t_c) t_c^k | p(x) \rangle x^k t_c^k.$$

In case $p(x) = [x]_n$ we obtain

$$[\sigma_q]_n = (q - 1)^n q^{\binom{n}{2}} x^n t_c^n. \tag{5.8}$$

We shall use this formula later.

2. The Sheffer sequence for $(\varepsilon_{y,c}(t_c)^{-1}, t_c)$ is

$$\begin{aligned} H_n(x; y) &= \varepsilon_{y,c}(t_c) x^n \\ &= \sum_{k=0}^n \binom{n}{k}_q y^{n-k} x^k. \end{aligned}$$

The polynomials $H_n(x; 1)$ are known as the q -Hermite polynomials. We have

$$\sum_{k=0}^{\infty} \frac{H_k(x; y)}{c_k} t^k = \varepsilon_{y,c}(t) \varepsilon_{x,c}(t)$$

and for $|q| < 1$, $|t| < 1/(1 - q)$,

$$\sum_{k=0}^{\infty} \frac{H_k(x; y)}{c_k} t^k = \prod_{j=0}^{\infty} \frac{1}{(1 - (1 - q) q^j y t)(1 - (1 - q) q^j x t)}$$

Equation (5.1) is

$$H_n(x; y) - H_n(qx; y) = (1 - q^n) x H_{n-1}(x; y).$$

The Expansion Theorem gives

$$x H_n(x; y) = \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,c}(t_c)^{-1} t_c^k | x H_n(x; y) \rangle}{c_k} H_k(x; y).$$

But

$$\begin{aligned} &\langle \varepsilon_{y,c}(t_c)^{-1} t_c | x H_n(x; y) \rangle \\ &= \langle \partial_c \varepsilon_{y,c}(t_c)^{-1} t_c^k | H_n(x; y) \rangle \\ &= \left\langle \frac{c_k}{c_{k-1}} \varepsilon_{y,c}(t_c)^{-1} t_c^{k-1} - (qt)^k y \varepsilon_{y,c}(qt_c)^{-1} | H_n(x; y) \right\rangle \\ &= c_k \delta_{k,n+1} - y \langle \varepsilon_{y,c}(t_c)^{-1} t_c^k | H_n(qx; y) \rangle \end{aligned}$$

and so

$$\begin{aligned} x H_n(x; y) &= \sum_{k=0}^{n+1} \frac{1}{c_k} (c_k \delta_{k,n+1} - y \langle \varepsilon_{y,c}(t_c)^{-1} t_c^k | H_n(qx; y) \rangle) H_k(x; y) \\ &= H_{n+1}(x; y) - y \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,c}(t_c)^{-1} t_c^k | H_n(qx; y) \rangle}{c_k} H_k(x; y) \\ &= H_{n+1}(x; y) - y H_n(qx; y). \end{aligned}$$

Thus

$$H_{n+1}(x; y) = (x + y \sigma_q) H_n(x; y).$$

Also

$$\begin{aligned} x^n &= \sum_{k=0}^n \frac{\langle \varepsilon_{y,c}(t_c)^{-1} t_c^k | x^n \rangle}{c_k} H_k(x; y) \\ &= \sum_{k=0}^n \binom{n}{k}_q \langle \varepsilon_{y,c}(t_c)^{-1} | x^{n-k} \rangle H_k(x; y) \end{aligned}$$

which, using (5.7), gives

$$x^n = \sum_{k=0}^n \binom{n}{k}_q (-y)^{n-k} q^{\binom{n-k}{2}} H_k(x; y).$$

The Expansion Theorem also gives

$$\begin{aligned} H_n(x; z) &= \sum_{k=0}^n \frac{\langle \varepsilon_{y,c}(t_c) t_c^k | H_n(x; z) \rangle}{c_k} [x]_{y,k} \\ &= \sum_{k=0}^n \binom{n}{k}_q H_{n-k}(y; z) [x]_{y,k}. \end{aligned}$$

Writing out $H_n(x; z)$, observing that $H_{n-k}(y; z) = H_{n-k}(z; y)$ and setting $y = 1$ gives

$$\sum_{k=0}^n \binom{n}{k}_q z^{n-k} x^k = \sum_{k=0}^n \binom{n}{k}_q H_{n-k}(z; 1) [x]_k.$$

Now in this equation we may replace x by the operator σ_q and since the resulting equation holds for all z , we may replace z by x giving

$$\sum_{k=0}^n \binom{n}{k}_q x^{n-k} \sigma_q^k = \sum_{k=0}^n \binom{n}{k}_q H_{n-k}(x; 1) [\sigma_q]_k.$$

According to (6.8)

$$[\sigma_q]_k H_m(x; 1) = (q-1)^k q^{\binom{k}{2}} \frac{c_m}{c_{m-k}} x^k H_{m-k}(x; 1)$$

and so

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k}_q x^{n-k} \sigma_q^k H_m(x; 1) \\ &= \sum_{k=0}^n \binom{n}{k}_q \binom{m}{k}_q c_k (q-1)^k x^k H_{n-k}(x; 1) H_{m-k}(x; 1). \end{aligned}$$

It remains to evaluate the left side of this equation. For $n = 1$ we get

$$(x + \sigma_q) H_m(x; 1) = H_{m+1}(x; 1)$$

and for $n = 2$, observing that $\sigma_q x = qx\sigma_q$, we have

$$\begin{aligned} &\left[\binom{2}{0}_q x^2 + \binom{2}{1}_q x\sigma_q + \binom{2}{2}_q \sigma_q^2 \right] H_m(x; 1) \\ &= (x^2 + (1+q)x\sigma_q + \sigma_q^2) H_m(x; 1) \\ &= (x^2 + x\sigma_q + \sigma_q x + \sigma_q^2) H_m(x; 1) \\ &= (x + \sigma_q)^2 H_m(x; 1) \\ &= H_{m+2}(x; 1). \end{aligned}$$

One easily sees, by induction, that

$$\sum_{k=0}^n \binom{n}{k}_q x^{n-k} \sigma_q^k = (x + \sigma_q)^n$$

and so we finally have

$$H_{m+n}(x; 1) = \sum_{k=0}^n \binom{n}{k}_q \binom{m}{k}_q c_k (q-1)^k x^k H_{n-k}(x; 1) H_{m-k}(x; 1),$$

a result first established by Carlitz [5].

6. A RELATED UMBRAL CALCULUS

In this section we shall consider the umbral calculus defined by setting

$$d_n = \frac{c_n}{(-\alpha)^n q^{\binom{n}{2}}} = \frac{c_n}{\langle \varepsilon_0 | [X]_{\alpha, n} \rangle}$$

for $\alpha \neq 0$. Then

$$\frac{d_n}{d_{n-1}} = \frac{q^{1-n}}{-\alpha} \frac{c_n}{c_{n-1}}$$

and

$$\frac{d_n}{d_k d_{n-k}} = q^{k(k-n)} \binom{n}{k}_q.$$

The operator t_d satisfies

$$\begin{aligned} t_d x^n &= \frac{q^{1-n}}{-\alpha} t_c x^n \\ &= \frac{1}{-\alpha} \sigma_{q^{-1}} t_c x^n \end{aligned}$$

and so

$$t_d = \frac{1}{-\alpha} \sigma_{q^{-1}} t_c$$

and

$$t_d p(x) = \frac{q}{-\alpha} \frac{p(x/q) - p(x)}{x - qx}.$$

If $s_n(x)$ is Sheffer for $t_{\mathbf{d}}$ we have

$$s_n(x/q) - s_n(x) = (q^{-n} - 1) x s_{n-1}(x).$$

It is easy to see that

$$\sigma_{q^{-1}} = 1 + (1 - q^{-1}) \alpha x t_{\mathbf{d}}.$$

The operator $\partial_{\mathbf{d}}$ satisfies a Leibniz-type formula

$$\begin{aligned} \partial_{\mathbf{d}} f(t) g(t) &= -\frac{1}{\alpha} \sigma_{q^{-1}}^* \partial_{\mathbf{c}} f(t) g(t) \\ &= f(t/q) \partial_{\mathbf{d}} g(t) + q(t) \partial_{\mathbf{d}} f(t). \end{aligned}$$

The exponential series $\varepsilon_{y,\mathbf{d}}(t)$ satisfies

$$\begin{aligned} \varepsilon_{y,\mathbf{d}}(t/q) &= (1 + (1 - q^{-1}) \alpha y t) \varepsilon_{y,\mathbf{d}}(t), \\ \partial_{\mathbf{d}} \varepsilon_{y,\mathbf{d}}(t)^{-1} &= -y \varepsilon_{y,\mathbf{d}}(t/q)^{-1} \end{aligned}$$

and

$$\partial_{\mathbf{d}} \varepsilon_{y,\mathbf{d}}(t) \varepsilon_{z,\mathbf{d}}(t) = (y + z + (1 - q^{-1}) \alpha y z t) \varepsilon_{y,\mathbf{d}}(t) \varepsilon_{z,\mathbf{d}}(t).$$

Theorem 4.4 shows that

$$\begin{aligned} \varepsilon_{y,\mathbf{d}}(t) &= \frac{\langle \varepsilon_0 | \varepsilon_{x,\mathbf{c}}(yt) \rangle}{\varepsilon_{\alpha,\mathbf{c}}(yt)} \\ &= \frac{1}{\varepsilon_{\alpha,\mathbf{c}}(yt)}. \end{aligned}$$

For $|q| < 1$ and $|t| < 1/(1 - q)$ we have

$$\varepsilon_{y,\mathbf{d}}(t) = \prod_{j=0}^{\infty} (1 - (1 - q) q^j \alpha y t).$$

The connection between \mathbf{d} and \mathbf{c} can be seen for $\alpha = -1$ by observing that if q is replaced by q^{-1} in d_n one obtains c_n .

Results similar to those of Section 5 may be obtained here in a completely analogous manner. For example, the numbers

$$G'_n = \sum_{k=0}^n q^{k(k-n)} \binom{n}{k}_q$$

satisfy the recurrence

$$G'_n = 2G'_{n-1} - (1 - q^{-(n-1)}) G'_{n-2}.$$

A solution to the recurrence

$$s_{n+1}(x) = (x - b_n) s_n(x) - e_n s_{n-1}(x),$$

$s_0(x) = 1$, Sheffer for t_d , occurs when

$$b_n = q^{-n} b_0,$$

$$e_{n+1} = -\frac{d_{n+1}}{d_n} q^{-n} \alpha e_1$$

in which case the solution is Sheffer for $(g(t_d), t_d)$ where

$$g(t) = \varepsilon_{y,d}(t) \varepsilon_{z,d}(t)$$

with $y + z = b_0$ and $(q^{-1} - 1) yz = e_1$.

The main reason for introducing the present umbral calculus is not so much for the above results but rather for results obtained by relating this umbral calculus to the q -umbral calculus (which are to follow).

Let us turn next to Sheffer sequences.

3. The Sheffer sequence for $(\varepsilon_{y,d}(t_d), t_d)$ is

$$\begin{aligned} s_n(x; y) &= \varepsilon_{y,d}(t_d)^{-1} x^n \\ &= \varepsilon_{\alpha,c}(yt_d) x^n \\ &= \sum_{k=0}^n \frac{(\alpha y)^k}{c_k} t_d^k x^n \\ &= \sum_{k=0}^n \frac{d_n}{c_k d_{n-k}} (\alpha y)^k x^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k}_q (-y)^{n-k} q^{\binom{k}{2}} \binom{n}{k}_q x^k. \end{aligned}$$

In a manner similar to that of Example 1 we see that

$$s_n(x; y) = (x - y)(x - q^{-1}y) \cdots (x - q^{-(n-1)}y).$$

The generating function for $s_n(x; y)$ is

$$\begin{aligned} \sum_{k=0}^n \frac{s_k(x; y)}{d_k} t^k &= \frac{\varepsilon_{x,d}(t)}{\varepsilon_{y,d}(t)} \\ &= \frac{\varepsilon_{\alpha,c}(yt)}{\varepsilon_{\alpha,c}(xt)}. \end{aligned}$$

For $|q| < 1$ and $|t| < 1/(1 - q)$

$$\sum_{k=0}^n \frac{s_k(x; y)}{d_k} t^k = \prod_{j=0}^{\infty} \frac{1 - (1 - q) q^j \alpha x t}{1 - (1 - q) q^j \alpha y t}.$$

4. The Sheffer sequence for $(\varepsilon_{y,d}(t_d)^{-1}, t_d)$ is

$$\begin{aligned} G_n(x; y) &= \varepsilon_{y,d}(t_d) x^n \\ &= \sum_{k=0}^n \binom{n}{k}_q q^{k(k-n)} y^{n-k} x^k. \end{aligned}$$

These polynomials were introduced by Carlitz [5] for $y = 1$. The generating function is

$$\begin{aligned} \sum_{k=0}^n \frac{G_k(x; y)}{d_k} t^k &= \varepsilon_{y,d}(t) \varepsilon_{x,d}(t) \\ &= \frac{1}{\varepsilon_{\alpha,c}(yt) \varepsilon_{\alpha,c}(xt)} \end{aligned}$$

and for $|q| < 1, |t| < 1/(1 - q)$

$$\sum_{k=0}^n \frac{G_k(x; y)}{d_k} t^k = \prod_{j=0}^{\infty} (1 - (1 - q) q^j \alpha y t)(1 - (1 - q) q^j \alpha x t).$$

As in the q -Hermite case we find that

$$G_{n+1}(x; y) = (x + y \sigma_{q^{-1}}) G_n(x; y).$$

Now that we have two distinct umbral calculi we may consider Corollary 4.6. Since $t_c = -\alpha \sigma_q t_d$ we have $y = q$ and $u(\sigma_q) = -\alpha \sigma_q$. Thus

$$\begin{aligned} u(\sigma_q^*) \cdots u(q^{k-1} \sigma_q^*) \varepsilon_{z,d}(t) &= (-\alpha)^k q^{\binom{k}{2}} \sigma_q^* k \varepsilon_{z,d}(t) \\ &= (-\alpha)^k q^{\binom{k}{2}} \varepsilon_{z,d}(q^k t) \end{aligned}$$

and Corollary 4.6 becomes

$$\varepsilon_{x,d}(t) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{c_k} q^{\binom{k}{2}} [x]_{z,k} t^k \varepsilon_{z,d}(q^k t).$$

Now we divide both sides by $\varepsilon_{z,d}(t)$ and observe that for $|q| < 1, |t| < 1/(1 - q)$,

$$\begin{aligned} \frac{\varepsilon_{z,d}(q^k t)}{\varepsilon_{z,d}(t)} &= \prod_{j=0}^{\infty} \frac{1 - (1 - q) q^{j+k} \alpha z t}{1 - (1 + q) q^j \alpha z t} \\ &= \prod_{j=0}^{k-1} \frac{1}{1 - (1 - q) q^j \alpha z t} \end{aligned}$$

and so

$$\prod_{j=0}^{\infty} \frac{1 - (1 - q) q^j \alpha x t}{1 - (1 - q) q^j \alpha z t} = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{c_k} q^{\binom{k}{2}} [x]_{z,k} t^k \sum_{j=0}^{k-1} \frac{1}{1 - (1 - q) q^j \alpha z t}.$$

Replacing t by $t/(1 - q)$ and setting $\alpha = 1$ gives

$$\prod_{j=0}^{\infty} \frac{1 - q^j x t}{1 - q^j z t} = \sum_{k=0}^{\infty} \frac{(z/x; q)_k}{(z; q)_k} \frac{q^{\binom{k}{2}}}{(q; q)_k} (-x t)^k.$$

7. ANOTHER UMBRAL CALCULI

Let us take

$$e_n = \frac{c_n}{[a]_{\alpha,n}} = \frac{c_n}{\langle \varepsilon_a | [x]_{\alpha k n} \rangle}$$

where $a \neq 0$ and $a \neq \alpha$. Then

$$\frac{e_n}{e_{n-1}} = \frac{1}{a - \alpha q^{n-1}} \frac{c_n}{c_{n-1}},$$

$$\frac{e_n}{e_k e_{n-k}} = \frac{[a]_{\alpha,k} [a]_{\alpha,n-k}}{[a]_{\alpha,n}} \binom{n}{k}_q.$$

In the notation of basic hypergeometric series $[a]_{\alpha,n} = a^n (\alpha/a; q)_n$ and so

$$e_n = \frac{c_n}{a^n (\alpha/a; q)_n}$$

and

$$\frac{e_n}{e_n e_{n-k}} = \frac{(\alpha/a; q)_k (\alpha/a; q)_{n-k}}{(\alpha/a; q)_n} \binom{n}{k}_q.$$

Observing that

$$(a - \alpha \sigma_q) t_e x^n = \frac{e_n}{e_{n-1}} (a - \alpha \sigma^{n-1}) x^{n-1} = t_c x^n$$

we have

$$t_e = \frac{1}{a - \alpha \sigma_q} t_c.$$

According to Theorem 4.4

$$\begin{aligned} \varepsilon_{y,e}(t) &= \frac{\langle \varepsilon_a \mid \varepsilon_{x,c}(yt) \rangle}{\varepsilon_{\alpha,c}(yt)} \\ &= \frac{\varepsilon_{a,c}(yt)}{\varepsilon_{\alpha,c}(yt)}. \end{aligned}$$

For $|q| < 1$ and $|t| < 1/(1 - q)$

$$\varepsilon_{y,e}(t) = \prod_{j=0}^{\infty} \frac{1 - (1 - q) q^j \alpha y t}{1 - (1 - q) q^j a y t}.$$

We also remark that

$$\begin{aligned} \varepsilon_{y,e}(qt) &= \frac{\varepsilon_{ay,c}(qt)}{\varepsilon_{\alpha y,c}(qt)} \\ &= \frac{1 - (1 - q) a y t}{1 - (1 - q) \alpha y t} \varepsilon_{y,e}(t). \end{aligned}$$

Let us have more Sheffer sequences.

5. The Sheffer sequence for $(\varepsilon_{y,e}(t_e), t_e)$ is

$$\begin{aligned} r_n(x; y) &= \varepsilon_{y,e}(t_e)^{-1} x^n \\ &= \frac{\varepsilon_{\alpha,c}(y t_e)}{\varepsilon_{a,c}(y t_e)} x^n \\ &= \sum_{k=0}^{\infty} \frac{[\alpha]_{a,k}}{c_k} y^k t_e^k x^n \\ &= \sum_{k=0}^n \frac{e_n}{c_k e_{n-k}} [\alpha]_{a,k} y^k x^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k}_q \frac{[\alpha]_{a,n-k} [a]_{\alpha,k}}{[a]_{\alpha,n}} y^{n-k} x^k. \end{aligned}$$

The generating function is (cf. [3, p. 365, Eq. 9.4])

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{r_k(x; y)}{e_k} t^k &= \frac{\varepsilon_{x,e}(t)}{\varepsilon_{y,e}(t)} \\ &= \frac{\varepsilon_{a,c}(xt) \varepsilon_{\alpha,c}(yt)}{\varepsilon_{\alpha,c}(xt) \varepsilon_{a,c}(yt)}. \end{aligned}$$

For $|q| < 1, |t| < 1/(1 - q),$

$$\sum_{k=0}^{\infty} \frac{r_k(x; y)}{e_k} t^k = \prod_{j=0}^{\infty} \frac{(1 - (1 - q) q^j a y t)(1 - (1 - q) q^j \alpha x t)}{(1 - (1 - q) q^j y t)(1 - (1 - q) q^j \alpha x t)}.$$

Theorem 2.5 is

$$r_n(xy; z) = \sum_{k=0}^n \frac{e_n}{e_k e_{n-k}} r_k(x; z) y^k r_{n-k}(yz; z).$$

We also have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{r_k(x; y)}{e_k} t^k &= \frac{\varepsilon_{x,c}(at) \varepsilon_{y,c}(\alpha t)}{\varepsilon_{y,c}(at) \varepsilon_{x,c}(\alpha t)} \\ &= \sum_{j=0}^{\infty} \frac{[x]_{y,j}}{c_j} (at)^j \sum_{i=0}^{\infty} \frac{[y]_{x,i}}{c_i} (\alpha t)^i \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \binom{k}{j}_q \frac{\alpha^j \alpha^{k-j}}{[a]_{x,k}} [x]_{y,j} [y]_{x,k-j} \right) \frac{t^k}{e_k} \end{aligned}$$

and so

$$r_k(x; y) = \sum_{j=0}^k \binom{k}{j}_q \frac{\alpha^j \alpha^{k-j}}{[a]_{x,k}} [x]_{y,j} [y]_{x,k-j}.$$

6. The Sheffer sequence for $(\varepsilon_{y,e}(t_e)^{-1}, t_e)$ is

$$\begin{aligned} u_n(x; y) &= \varepsilon_{y,e}(t_e) x^n \\ &= \sum \frac{e_n}{e_k e_{n-k}} y^{n-k} x^k. \end{aligned}$$

The generating function is

$$\begin{aligned} \sum \frac{u_k(x; y)}{e_k} t^k &= \varepsilon_{y,e}(t) \varepsilon_{x,e}(t) \\ &= \frac{\varepsilon_{a,c}(y t) \varepsilon_{a,c}(x t)}{\varepsilon_{a,c}(y t) \varepsilon_{a,c}(x t)}. \end{aligned}$$

Next let us consider Corollary 4.6. Here we have $t_c = (a - \alpha \sigma_q) t_e$ and so $u(\sigma_q) = a - \alpha \sigma_q.$ Thus

$$u(\sigma_q^*) \cdots u(q^{k-1} \sigma_q^*) \varepsilon_{z,e}(t) = (a - \alpha \sigma_q^*) \cdots (a - \alpha q^{k-1} \sigma_q^*) \varepsilon_{z,e}(t).$$

For practice we take $k = 1$,

$$\begin{aligned} & (a - \alpha\sigma_q^*) \varepsilon_{z,e}(t) a \varepsilon_{z,e}(t) - \alpha \varepsilon_{z,e}(qt) \\ &= \left[a - \alpha \frac{1 - (1 - q) azt}{1 - (1 - q) \alpha zt} \right] \varepsilon_{z,e}(t) \\ &= \frac{a - \alpha}{1 - (1 - q) \alpha zt} \varepsilon_{z,e}(t). \end{aligned}$$

For $k = 2$ we have

$$\begin{aligned} & (a - \alpha\sigma_q^*)(a - \alpha q\sigma_q^*) \varepsilon_{z,e}(t) \\ &= (a - \alpha q \varepsilon_q^*) \frac{a - \alpha}{1 - (1 - q) \alpha zt} \varepsilon_{z,e}(t) \\ &= \left[a \frac{a - \alpha}{1 - (1 - q) \alpha zt} \right. \\ &\quad \left. - \alpha q \frac{a - \alpha}{1 - (1 - q) \alpha zqt} \frac{1 - (1 - q) azt}{1 - (1 - q) \alpha zt} \right] \varepsilon_{z,e}(t) \\ &= \frac{a - \alpha}{1 - (1 - q) \alpha zt} \frac{a - \alpha q}{1 - (1 - q) \alpha zqt} \varepsilon_{z,e}(t). \end{aligned}$$

From here it is an easy matter to see (by induction) that

$$\begin{aligned} & (a - \alpha\sigma_q^*) \cdots (a - \alpha q^{k-1} \sigma_q^*) \varepsilon_{z,e}(t) \\ &= \frac{(a - \alpha) \cdots (a - \alpha q^{k-1})}{(1 - (1 - q) \alpha zt) \cdots (1 - (1 - q) \alpha zq^{k-1} t)} \varepsilon_{z,e}(t) \\ &= [a]_{\alpha,k} \varepsilon_{z,e}(t) \prod_{j=0}^{k-1} \frac{1}{1 - (1 - q) \alpha zq^j t}. \end{aligned}$$

Thus Corollary 4.6 becomes

$$\varepsilon_{x,e}(t) = \sum_{k=0}^{\infty} \frac{[a]_{\alpha,k}}{c_k} [x]_{z,k} t^k \left[\prod_{j=0}^{k-1} \frac{1}{1 - (1 - q) \alpha zq^j t} \right] \varepsilon_{z,e}(t).$$

Dividing both sides by $\varepsilon_{z,e}(t)$ and replacing t by $t/(1 - q)$ gives

$$\begin{aligned} \prod_{j=0}^{\infty} \frac{(1 - q^j azt)(1 - q^j \alpha xt)}{(1 - q^j azt)(1 - q^j \alpha xt)} &= \sum_{k=0}^{\infty} \frac{(\alpha/a; q)_k (z/x; q)_k}{(\alpha zt; q)_k (q; q)_k} (axt)^k \\ &= {}_2\phi_1(\alpha/a; z/x; \alpha zt, q; axt). \end{aligned}$$

This is the basic analog of Gauss' theorem [11, p. 97].

8. SOME ADDITIONAL EXPANSIONS

In this section we shall very briefly consider some expansions, including Carlitz q -analog of the Lagrange inversion theorem [4]. A more thorough discussion must await a sequel to this paper. Let $c_n = (1 - q) \cdots (1 - q^n) / (1 - q)^n$ be as in Section 5.

Recalling the generating function for $[x]_{y,c}$ we begin by observing that

$$\begin{aligned} &\langle \varepsilon_{y,c}(q^{-n}t_c) t_c^n \mid \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k \rangle \\ &= \frac{c_k}{c_{k-n}} \left\langle \frac{\varepsilon_{y,c}(q^{-n}t_c)}{\varepsilon_{y,c}(q^{-k+1}t_c)} \mid x^{k-n} \right\rangle \\ &= \frac{c_k}{c_{k-n}} \left\langle \sum_{j=0}^{\infty} \frac{[yq^{-n}]_{yq^{-k+1},j}}{c_j} \langle t_c^j \mid x^{k-n} \rangle \right\rangle \\ &= \frac{c_k}{c_{k-n}} [yq^{-n}]_{yq^{-k+1},k-n} \\ &= c_n \delta_{n,k}. \end{aligned} \tag{8.1}$$

From this we obtain the expansion

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) \mid \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k \rangle}{c_k} \varepsilon_{y,c}(q^{-k}t) t^k \tag{8.2}$$

valid for all series $f(t)$, since it holds $f(t) = \varepsilon_{y,c}(q^{-n}t) t^n$ for all $n \geq 0$. Variations on this expansion are possible since for any invertible $g(t)$ we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) \mid g(t_c) \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k \rangle}{c_k} g(t)^{-1} \varepsilon_{y,c}(q^{-k}t) t^k.$$

Also, from (5.7) we get

$$\begin{aligned} \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k &= \sum_{j=0}^{\infty} \frac{(-yq^{-k+1})}{c_j} q^{\binom{j}{2}} t_c^j x^k \\ &= \sum_{j=0}^k \binom{k}{j}_q (-yq^{-k+1})^j q^{\binom{j}{2}} x^{k-j} \end{aligned}$$

and so (8.2) can be written

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{c_k} \left[\sum_{j=0}^k \binom{k}{j}_q (-yq^{-k+1})^j q^{\binom{j}{2}} \langle f(t_c) \mid x^{k-j} \rangle \right] \varepsilon_{y,c}(q^{-k}t) t^k.$$

As a simple example, if we set $f(t) = \varepsilon_{0,c}(t) = 1$ then we get

$$1 = \sum_{k=0}^{\infty} \frac{(-y)^k}{c_k} q^{-\binom{k}{2}} \varepsilon_{y,c}(q^{-k}t) t^k.$$

Setting $y = 1/(1-q)$, multiplying by $\varepsilon_{y,c}(t)^{-1}$ and observing that $\varepsilon_{y,c}(q^{-k}t)/\varepsilon_{y,c}(t) = 1/(1 - (1-q)tq^{-k}) \cdots (1 - (1-q)tq^{-1})$ we have (after replacing t by $t/(1-q)$),

$$\prod_{j=0}^{\infty} (1 - tq^j) = \sum_{k=0}^{\infty} \frac{(-1)^k}{c_k(1-q)^k} q^{-\binom{k}{2}} \frac{t^k}{(1-tq^{-k}) \cdots (1-tq^{-1})}.$$

In Slater's [11] notation this is

$$\prod_{j=0}^{\infty} (1 - tq^j) = \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2}} \frac{t^k}{(q^{-k}t; q)_k (q; q)_k}.$$

Incidentally, in the terminology of [9] the sequence $s_k(x) = \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k$ is known as a decentralized Sheffer sequence. Moreover, Theorem 4.4 of [9] gives a formula which shows that if $\langle \varepsilon_{y,c}(q^{-n}t_c) t_c^n | s_k(x) \rangle = c_n \delta_{n,k}$ then $s_k(x) = \varepsilon_{y,c}(q^{-k+1}t_c)^{-1} x^k$. Thus the sequence $s_k(x)$ was not obtained by sheer guesswork. But since we assume no familiarity with [9] we have given the derivation (8.1) above.

In a similar way we may verify that

$$\langle \varepsilon_{y,c}(q^n t_c)^{-1} t_c^n | \varepsilon_{y,c}(q^{k-1} t_c) x^k \rangle = c_n \delta_{n,k}.$$

Thus for any $f(t)$ and invertible $g(t)$,

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) | g(t_c) \varepsilon_{y,c}(q^{k-1} t_c) x^k \rangle}{c_k} g(t)^{-1} \varepsilon_{y,c}(q^k t)^{-1} t^k. \quad (8.3)$$

As an example, let us take $g(t) = \varepsilon_{y,c}(t)^{-1}$. Then

$$\begin{aligned} \frac{\varepsilon_{y,c}(q^m t)}{\varepsilon_{y,c}(t)} &= \prod_{j=0}^{m-1} (1 - (1-q) q^j y t) \\ &= ((1-q) y t; q)_m \end{aligned}$$

and taking $y = 1/(1-q)$ gives

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) | (t; q)_{k-1} x^k \rangle}{c_k} \frac{t^k}{(t; q)_k}.$$

It is easy to verify that

$$\frac{\varepsilon_{y,c}(q^{k-1}t_c)}{\varepsilon_{y,c}(t_c)} x^k = x \frac{\varepsilon_{y,c}(q^k t_c)}{\varepsilon_{y,c}(t_c)} x^{k-1}.$$

Thus for $g(t) = \varepsilon_{y,c}(t)^{-1}$ Eq. (8.3) is

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) | x(t_c; q)_k x^{k-1} \rangle}{c_k} \frac{t^k}{(t; q)_k}. \tag{8.4}$$

This is Eq. (1.11) of [4], which is Carlitz' *q*-analog of a special case of the Lagrange inversion formula. To see this we note that

$$\begin{aligned} \langle f(t_c) | x(t_c; q)_k x^{k-1} \rangle &= \langle \partial_c f(t_c) | (t_c; q)_k x^{k-1} \rangle \\ &= \langle (t_c; q)_k \partial_c f(t_c) | x^{k-1} \rangle \\ &= \langle \partial_c^{k-1} [(t_c; q)_k \partial_c f(t_c)] | x^0 \rangle. \end{aligned}$$

In the notation of [4] this is $(1-q)^{-k} \Delta_0^{k-1} [(t)_k A f(t)]$ and since $c_k = (q)_k (1-q)^k$ we get (1.11). Similar considerations would lead to other *q*-expansions, such as (3.14) of [4].

REFERENCES

1. W. R. ALLAWAY, Some properties of the *q*-Hermite polynomials, *Canad. J. Math.* **32** (1980), 686-694.
2. W. AL-SALAM, *q*-Appell polynomials, *Ann. Mat. Pura Appl. (4)* **77** (1967), 31-46.
3. G. ANDREWS, On the foundations of combinatorial theory. V. Eulerian differential operators, in "Studies in Applied Mathematics, Vol. L, No. 4," pp. 345-375, 1971.
4. L. CARLITZ, Some *q*-expansion formulas, *Glas. Mat.* **8**, No. (28) (1973), 205-214.
5. L. CARLITZ, Some polynomials related to theta functions, *Duke J. Math.* **24** (1957), 521-527.
6. J. GOLDMAN AND G.-C. ROTA, The number of subspaces of a vector space, in "Recent Progress in Combinatorics," Academic Press, New York, 1969.
7. J. GOLDMAN AND G.-C. ROTA, On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions, in "Studies in Applied Mathematics, Vol. XLIX, No.3," pp. 239-257, 1970.
8. S. ROMAN, The theory of the umbral calculus, I, *J. Math. Anal. Appl.* **87** (1982), 57-115.
9. S. ROMAN, The theory of the umbral calculus, II, *J. Math. Anal. Appl.* **89** (1982), 290-314.
10. S. ROMAN, The theory of the umbral calculus, III, *J. Math. Anal. Appl.* **95** (1983), 528-563.
11. L. SLATER, "Generalized Hypergeometric Functions," Cambridge Univ. Press, Cambridge, 1966.

Additional references—this list is by no means meant to be complete.

W. R. ALLAWAY, A comparison of two umbral algebras, *J. Math. Anal. Appl.* **85** (1982), 197-235.

W. R. ALLAWAY AND K. W. YUEN, Ring isomorphisms for the family of Eulerian differential operators, *J. Math. Anal. Appl.* **77** (1980), 245-263.

- A. M. CHAK, An expansion of a class of polynomials, I, *Riv. Mat. Univ. Parma* (2) **12** (1971), 47–55.
- A. M. CHAK, An extension of a class of polynomials, II, *SIAM J. Math. Anal.* **2** (1971), 352–355.
- A. M. CHAK, An extension of a class of polynomials, III, *Riv. Mat. Univ. Parma* (3) **2** (1973), 11–18.
- E. C. IHRIG AND M. E. H. ISMAIL, A q -numbral calculus, *J. Math. Anal. Appl.* **84** (1981), 89–93.
- D. L. REINER, Eulerian binomial type revisited, *Stud. Appl. Math.* **64** (1981), 89–93.
- S. ROMAN, The algebra of formal series, *Adv. in Math.* **31** (1979), 309–329. Erratum **35** (1980), 274.
- S. ROMAN, The algebra of formal series. II. Sheffer sequences, *J. Math. Anal. Appl.* **74** (1980), 120–143.
- S. ROMAN, The algebra of formal series, III, *J. Approx. Theory* **26** (1979), 340–381.
- S. ROMAN AND G.-C. ROTA, The umbral calculus, *Adv. in Math.* **27** (1978), 95–188.
- G.-C. ROTA, D. KAHARER, AND A. ODLYZKO, Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 685–760.
- A. SHARMA AND A. M. CHAK, The basic analog of a class of polynomials, *Riv. Mat. Univ. Parma* **5** (1954), 325–337.
- M. WARD, A calculus of sequences, *Amer. J. Math.* **58** (1936), 255–266.