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More on the Umbral Calculus, with Emphasis on the q-Umbral Calculus

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The interrelationship between distinct umbral calculi is studied. These ideas are applied in particular to q-umbral calculus, which shows how Andrews' q-theory relates to the q-theory discussed in the previous paper by the author. The q-Hermite polynomials and basic hypergeometric series are briefly discussed. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this paper we continue the study of the umbral calculus begun in [8] (see also [9, 10]). Some knowledge of Sections 1–5 and 11 of [8] would be helpful here.

Recall that for each sequence c_n of non-zero constants we defined a distinct umbral calculus, used in the study of polynomial sequences $s_n(x)$ whose generating function has the form

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} t^k = \frac{1}{g(\bar{f}(t))} \varepsilon_x(\bar{f}(t))$$

where $g(t) = g_0 + g_1 t + \cdots (g_0 \neq 0)$, $f(t) = f_1 t + f_2 t^2 + \cdots (f_1 \neq 0)$ and

$$\varepsilon_x(t) = \sum_{k=0}^{\infty} \frac{x^k}{c_k} t^k.$$

We called these sequences $s_n(x)$ Sheffer sequences. In Section 11 of [8] we touched briefly on the q-umbral calculus, defined by

$$c_n = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

Our objective in this paper is to begin studying the relationship between

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distinct umbral calculi whose defining sequences c_n are related. We have restricted attention to the case $\bar{f}(t)=t$.

The first step, in Section 2, is to extend slightly the notion of Sheffer sequence (for $\bar{f}(t)=t$) to include sesquences $r_n(x)$ for which

$$\sum_{k=0}^{\infty} \frac{r_k(x)}{d_k} t^k = \frac{1}{g(t)} \varepsilon_x(t) \tag{1.1}$$

where $g(t)$ and $\varepsilon_x(t)$ are as above. Notice that the sequence d_n on the left may be different from the sequence c_n used in defining $\varepsilon_x(t)$. While this extension is modest from one point of view, namely that $(d_n/c_n)\, r_n(x)$ is Sheffer in the c_n-umbral calculus, it does serve a useful purpose. For example, the q-umbral calculus in [8] employs the identity

$$\varepsilon_y(t)\, s_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q s_k(x)\, s_{n-k}(y)$$

whereas Andrews' q-theory [3] employs the identity

$$s_n(xy) = \sum_{k=0}^{n} \binom{n}{k}_q s_k(x)\, y^k s_{n-k}(xy).$$

Clarification of the roles of these identities is made possible by the aforementioned extension (see Theorem 2.5). We call sequences $r_n(x)$ satisfying (1.1) basic sequences, although this term is a bit overused.

In Section 3 we consider a class of operators whose importance in the umbral calculus was recognized by Andrews. In Section 4 we study the relationship between the exponential-type series $\varepsilon_x(t)$ for different umbral calculi. It is this series which lies at the heart of the theory.

Sections 5–7 are devoted to the q-umbral calculus and related umbral calculi. The q-Hermite polynomials, among others, are placed in the context of the umbral calculus. Infinite products are discussed briefly. Heine's theorem [11, p. 92] is shown to be nothing but a generating function and the basic analog of Gauss' theorem [11, p. 97] is shown to be but a special case of a result from Section 4. Space limitations force us to postpone to a sequel a more detailed discussion of these umbral calculi or of others.

In Section 8 we discuss very briefly some additional expansion formulas, including Carlitz's q-analog of a special case of the Lagrange inversion formula.

We wish to express our indebtedness to Andrews' work [3]. The influence of his ideas is clearly visible in this paper.

2. Basic Results

A sequence c_n of constants is *admissible* if $c_n \neq 0$ for all $n \geqslant 0$. We shall use boldface type to denote admissible sequences—$\mathbf{c}$ for c_n, $\mathbf{d}$ for d_n, etc.

For each admissible sequence $\mathbf{c}$ we define linear functionals and operators, as in [8], by

$$\langle t_{\mathbf{c}}^k \mid x^n \rangle = c_n \delta_{n,k},$$

$$\begin{aligned} t_{\mathbf{c}}^k x^n &= \frac{c_n}{c_{n-k}} x^{n-k}, & k \leqslant n, \\ &= 0, & n < k. \end{aligned}$$

The subscript $\mathbf{c}$ is needed since we shall be dealing with more than one admissible sequence at a time.

We recall from [8] that the sequence $s_n(x)$ is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$, where $g(t)$ is invertible, if and only if any one of the following equivalent conditions holds:

(1) $\langle g(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid s_n(x) \rangle = c_n \delta_{n,k}$,

(2) $\langle g(t_{\mathbf{c}}) \mid s_n(x) \rangle = c_0 \delta_{n,0}$, $t_{\mathbf{c}} s_n(x) = (c_n/c_{n-1})\, s_{n-1}(x)$,

(3) $\sum_{k=0}^{\infty} (s_k(x)/c_k)\, t^k = \varepsilon_{x,c}(t)/g(t)$, where $\varepsilon_{x,\mathbf{c}}(t) = \sum_{k=0}^{\infty} (x^k/c_k)\, t^k$.

Notice that (3) is a formal equation in the formal variables x and t and so there is no need for a subscript on t. We say that $s_n(x)$ is *Sheffer for* $t_{\mathbf{c}}$ if it is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$ for some invertible $g(t)$.

Let $\mathbf{c}$ and $\mathbf{d}$ be admissible sequences and let L be a linear functional for which $\langle L \mid 1 \rangle \neq 0$. We say that the sequence $p_n(x)$ is the *basic sequence* for $(t_{\mathbf{c}}, \mathbf{d}, L)$ if

(1) $\deg p_n(x) = n$,

(2) $\langle L \mid p_n(x) \rangle = d_0 \delta_{n,0}$,

(3) $t_{\mathbf{c}}\, p_n(x) = (d_n/d_{n-1})\, p_{n-1}(x)$.

For each choice of sequence c the linear functional L has a series reresentation $g_{L,\mathbf{c}}(t_{\mathbf{c}}) = \sum_{k=0}^{\infty} [\langle L \mid x^k \rangle / c_k]\, t_{\mathbf{c}}^k$. Thus as linear functionals $L = g_{L,\mathbf{c}}(t_{\mathbf{c}})$. Note that $g_{L,\mathbf{c}}(t)$ is invertible since $\langle L \mid 1 \rangle \neq 0$.

The following theorem is immediate from the definitions.

Theorem 2.1. *The following are equivalent.*

(i) $p_n(x)$ *is basic for* $(t_{\mathbf{c}}, \mathbf{d}, L)$,

(ii) $s_n(x) = (c_n/d_n)\, p_n(x)$ *is Sheffer for* $(g_{L,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$,

(iii) $\langle g_{L,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid p_n(x) \rangle = d_n \delta_{n,k}$,

(iv) $p_n(x) = (d_n/c_n)\, g_{L,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, x^n$,

(v) $\sum_{k=0}^{\infty} (p_k(x)/d_k)\, t^k = \varepsilon_{y,\mathbf{c}}(t)/g_{L,\mathbf{c}}(t)$.

Part (iii) of Theorem 2.1 gives us a version of the Expansion Theorem and its important corollary.

THEOREM 2.2. *If* $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$ *then*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t_{\mathbf{c}}) \mid p_k(x) \rangle}{d_k}\, g_{L,\mathbf{c}}(t)\, t^k.$$

THEOREM 2.3. *If* $p_n(x)$ *is basic for* $(t_{\mathbf{c}}, \mathbf{d}, L)$ *then*

$$p(x) = \sum_{k \geqslant 0} \frac{\langle g_{L,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid p(x) \rangle}{d_k}\, p_k(x).$$

A useful way to organize basic sequences is as follows. For each linear functional L satisfying $\langle L \mid 1 \rangle \neq 0$ we define the *L-table* or *L-matrix* to be the matrix whose rows and columns are indexed by the $2^{\aleph_0}$ possible admissible sequences and whose $(\mathbf{c}, \mathbf{d})$ entry is the basic sequence for $(t_{\mathbf{c}}, \mathbf{d}, L)$, which we may denote by $B_{\mathbf{c},\mathbf{d}}(L)$. In L-table language, part (i) of Theorem 2.1 says that those entries of an L-table which are Sheffer sequences are precisely the diagonal entries $B_{\mathbf{c},\mathbf{c}}(L)$. According to part (ii) of this theorem, entries in the same row of an L-table are related in a simple manner. Borrowing notation from the old style umbral calculus we may write

$$B_{\mathbf{c},\mathbf{e}}(L) = \frac{\mathbf{e}}{\mathbf{d}}\, B_{\mathbf{c},\mathbf{d}}(L).$$

More exactly, if $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{e}, L)$ and $r_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$ then

$$p_n(x) = \frac{e_n}{d_n}\, r_n(x).$$

The columns of an L-table present more of a challenge than the rows. It is our intention to give an algebraic identity which characterizes these columns. We begin with a result of Andrews [3], in a slightly different form, which characterizes the operators $t_{\mathbf{c}}$. Let σ_y be the linear operator on polynomials defined by

$$\sigma_y\, p(x) = p(xy).$$

THEOREM 2.4. *Let τ be a linear operator on polynomials. Then $\tau = t_{\mathbf{c}}$ for some admissible sequence* $\mathbf{c}$ *if and only if*

$$\tau x^n \neq 0 \qquad \textit{all } n > 0$$

$$\tau\sigma_y = y\sigma_y\tau$$

for all constants y.

Proof. One easily sees that $t_{\mathbf{c}}\sigma_y x^n = y\sigma_y t_{\mathbf{c}} x^n$ and $t_{\mathbf{c}} x^n \neq 0$ for $n > 0$. For the converse, set $\tau x^n = s_n(x)$. Then $ys_n(xy) = y\sigma_y s_n(x) = y\sigma_y \tau x^n = \tau\sigma_y x^n = y^n \tau x^n = y^n s_n(x)$. Setting $x = 1$ and changing y to x gives

$$\tau x^n = s_n(x) = s_n(1)\, x^{n-1}, \qquad n > 0,$$

$$\tau 1 = s_0(x) = 0.$$

Since $s_n(1) \neq 0$ (otherwise $\tau x^n = 0$) we may set

$$c_n = s_1(1)\, s_2(1) \cdots s_n(1)\, c_0$$

with $c_0 \neq 0$ arbitrary. Then $\tau x^n = (c_n/c_{n-1})\, x^{n-1} = t_{\mathbf{c}} x^n$. Thus $\tau = t_{\mathbf{c}}$.

We are now ready to characterize the columns of an L-table.

THEOREM 2.5. *The sequence $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$ for some admissible sequence* $\mathbf{c}$ *if and only if* $\deg p_n(x) = n$ *and*

$$p_n(xy) = \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}}\, p_k(x)\, y^k \langle L \mid p_{n-k}(xy)\rangle \tag{2.1}$$

for all constants y. In other words, $p_n(x)$ is in the $\mathbf{d}$th *column of the L-table if and only if it satisfies* (2.1).

Proof. First assume $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$. The Expansion Theorem gives

$$p_n(xy) = \sum_{k=0}^{n} \frac{p_k(xy)}{c_k} \langle g_{L,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid p_n(xy)\rangle.$$

But

$$\begin{aligned}\langle g_{L,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid p_n(xy)\rangle &= \langle g_{L,\mathbf{c}}(t_{\mathbf{c}}) \mid t_{\mathbf{c}}^k \sigma_y p_n(x)\rangle \\ &= y^k \langle g_{L,\mathbf{c}}(t_{\mathbf{c}}) \mid \sigma_y t_{\mathbf{c}}^k p_n(x)\rangle \\ &= \frac{d_n}{d_{n-k}}\, y^k \langle L \mid \sigma_y p_{n-k}(x)\rangle\end{aligned}$$

and the result follows. For the converse, suppose $\deg p_n(x) = n$ and $p_n(x)$

satisfies (2.1). Then let τ be the linear operator defined by $\tau p_n(x)=(d_n/d_{n-1})\,p_{n-1}(x)$. Now (2.1) can be written as

$$\sigma_y p_n(x)=\sum_{k=0}^{n}\frac{d_n}{d_k d_{n-k}}\,p_{n-k}(x)\,y^{n-k}\langle L \mid p_k(xy)\rangle$$

and applying τ gives

$$\begin{aligned}\tau\sigma_y p_n(x)&=\sum_{k=0}^{n-1}\frac{d_n}{d_k d_{n-k}}\frac{d_{n-k}}{d_{n-k-1}}\,p_{n-k-1}(x)\,y^{n-k}\langle L \mid p_k(xy)\rangle\\ &=y\frac{d_n}{d_{n-1}}\sum_{k=0}^{n-1}\frac{d_{n-1}}{d_k d_{n-1-k}}\,p_{n-1-k}(x)\,y^{n-1-k}\langle L \mid p_k(xy)\rangle\\ &=y\frac{d_n}{d_{n-1}}\,\sigma_y\,p_{n-1}(x)\\ &=y\sigma_y\tau p_n(x).\end{aligned}$$

Thus $\tau\sigma_y=y\sigma_y\tau$ and since $\tau x^n\neq 0$ we deduce from Theorem 2.4 the existence of an admissible sequence $\mathbf{c}$ for which $\tau=t_{\mathbf{c}}$. Thus $t_{\mathbf{c}}p_n(x)=(d_n/d_{n-1})\,p_{n-1}(x)$. Finally, setting $y=1$ in (2.1) and comparing coefficients of $p_k(x)$ on both sides gives $\langle L \mid p_n(x)\rangle=d_0\delta_{n,0}$. Hence $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$ and the proof is complete.

Let us consider Andrews' q-theory in the present context. Comparing his Definition 1 in [3, p. 349] with (2.1) one sees that his theory is devoted to the dth column of the ε_1 table where ε_1 is evaluation at $x=1$ and $d_n=(1-q)\cdots(1-q^n)/(1-q)^n$. Incidentally, this explains why there is no transfer formula [8, Eq. (7.1)] in Andrews' q-theory, for this formula reduces to triviality when $\bar{f}(t)=t$. Further, there is a recurrence formula for basic sequences, obtained directly from [8, Eq. (6.3)] but unfortunately it seems difficult to employ usefully in the present context.

3. Invariant Operators

Let us call a linear operator τ *invariant* if

$$\tau\sigma_y=\sigma_y\tau$$

for all constants y. Andrews [3] used the term Eulerian shift-invariant. Some examples of invariant operators are:

(1) σ_y and any linear combination of such operators,

(2) $xt_{\mathbf{c}}$ and any polynomial in $xt_{\mathbf{c}}$,

(3) $x^k t_{\mathbf{c}}^k$.

With regard to the last example, we state without proof (which follows the lines of that for Theorem 2.4) the following result.

THEOREM 3.1. *Let τ be invariant and suppose $\tau x^n \neq 0$ for $k \leqslant n$ but $\tau x^n = 0$ for $n < k$. Then $\tau = x^k t_{\mathbf{c}}^k$ for some* $\mathbf{c}$.

Invariant operators have their own Expansion Theorem. If L is a linear functional then by L_y we mean the linear operator acting on polynomials in x and y defined by

$$L_y x^n y^m = \langle L \mid x^m \rangle \, x^n.$$

We also define the operator $\bar{L}$, acting on polynomials in x, by

$$\bar{L} x^n = \langle L \mid x^n \rangle \, x^n.$$

Notice that $\bar{L} x^n = L_y x^n y^n = L_y \sigma_y x^n$ and so $\bar{L} = L_y \sigma_y$.

THEOREM 3.2. *Let τ be invariant and suppose that $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, L)$. Then*

$$\tau = \sum_{k=0}^{\infty} \frac{1}{d_k} \langle \varepsilon_1 \mid \tau p_k(x) \rangle \, x^k \bar{L} t_{\mathbf{c}}^k$$

where ε_1 is evaluation at $x = 1$.

Proof. We first observe that $\tau \sigma_y p_n(x) = \sigma_y \tau p_n(x)$ is symmetric in x and y and so, treating y as a variable, we have $\tau \sigma_y p_n(x) = \tau_y (\sigma_x)_y \, p_n(y) = \tau_y p_n(xy)$. Using Eq. (2.1) with x and y interchanged we have

$$\begin{aligned}
\tau p_n(x) &= \langle (\varepsilon_1)_y \mid \sigma_y \tau p_n(x) \rangle \\
&= \langle (\varepsilon_1)_y \mid \tau_y p_n(xy) \rangle \\
&= \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} \langle (\varepsilon_1)_y \mid \tau_y p_k(y) \rangle \, x^k L_y p_{n-k}(xy) \\
&= \sum_{k=0}^{n} \frac{d_n}{d_k d_{n-k}} \langle \varepsilon_1 \mid \tau p_k(x) \rangle \, x^k L_y \sigma_y p_{n-k}(x) \\
&= \sum_{k=0}^{n} \frac{1}{d_k} \langle \varepsilon_1 \mid \tau p_k(x) \rangle \, x^k L_y \sigma_y t_{\mathbf{c}}^k p_n(x).
\end{aligned}$$

The result follows.

Since $\bar{L}$ is the identity when $L = \varepsilon_1$ we obtain the following corollaries.

COROLLARY 3.3. *Let τ be invariant and suppose $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, \varepsilon_1)$. Then*

$$\tau = \sum_{k=0}^{\infty} \frac{1}{d_k} \langle \varepsilon_1 \mid \tau p_k(x) \rangle \, x^k t_{\mathbf{c}}^k.$$

COROLLARY 3.4. *If $p_n(x)$ is basic for $(t_{\mathbf{c}}, \mathbf{d}, \varepsilon_1)$ then*

$$\sigma_z = \sum_{k=0}^{\infty} \frac{1}{d_k} p_k(z) \, x^k t_{\mathbf{c}}^k.$$

Before we leave this section we observe that

$$\begin{aligned}\langle \sigma_y^* t_{\mathbf{c}}^k \mid x^n \rangle &= \langle t_{\mathbf{c}}^k \mid \sigma_y x^n \rangle \\ &= y^n c_n \delta_{n,k} \\ &= \langle (y t_{\mathbf{c}})^k \mid x^n \rangle\end{aligned}$$

and so

$$\sigma_y^* f(t) = f(yt).$$

In other symbols.

$$\langle f(y t_{\mathbf{c}}) \mid p(x) \rangle = \langle f(t_{\mathbf{c}}) \mid p(xy) \rangle.$$

4. EXPONENTIAL-TYPE SERIES

In this section we shall consider the series

$$\varepsilon_{y,\mathbf{c}}(t) = \sum_{k=0}^{\infty} \frac{y^k}{c_k} t^k.$$

This is the "exponential" series in the $\mathbf{c}$-umbral calculus. We define the linear operator $\partial_{\mathbf{c}}$ acting on power series in t by

$$\partial_{\mathbf{c}} t^k = \frac{c_k}{c_{k-1}} t^{k-1}.$$

Thus, for example, $\langle \partial_{\mathbf{c}} t_{\mathbf{d}}^k \mid x^n \rangle = (c_k/c_{k-1}) \langle t_{\mathbf{d}}^{k-1} \mid x^n \rangle = (c_k/c_{k-1}) d_{k-1} \delta_{k-1,n}$. It is easy to see [take $f(t) = t^k$ and $p(x) = x^n$] that

$$\langle f(t_{\mathbf{c}}) \mid x p(x) \rangle = \langle \partial_{\mathbf{c}} f(t_{\mathbf{c}}) \mid p(x) \rangle.$$

Also

$$\partial_{\mathbf{c}} \varepsilon_{y,\mathbf{c}}(t) = y \in_{y,\mathbf{c}}(t).$$

Our next goal is to get some information about $\partial_{\mathbf{d}} \varepsilon_{y,\mathbf{c}}(t)$. One easily verifies the next lemma by setting $p(x) = x^n$.

LEMMA 4.1. $\langle t_{\mathbf{c}} \partial_{\mathbf{d}} \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}}) \mid p(x) \rangle = \langle \varepsilon_1 \mid t_{\mathbf{d}} p(xy) \rangle$.

THEOREM 4.2. *Let $s_n(x)$ be Sheffer for $(\varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$. Then*

$$\partial_{\mathbf{d}} \varepsilon_{y,\mathbf{c}}(t) = \left(\sum_{k=0}^{\infty} \frac{1}{c_{k+1}} \langle \varepsilon_1 \mid t_{\mathbf{d}} s_{k+1}(xy) \rangle t^k \right) \varepsilon_{y,\mathbf{c}}(t).$$

Proof. By the Expansion Theorem

$$\begin{aligned} t \partial_{\mathbf{d}} \varepsilon_{y,\mathbf{c}}(t) &= \sum_{k=1}^{\infty} \frac{1}{c_k} \langle t_{\mathbf{c}} \partial_{\mathbf{d}} \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}}) \mid s_k(x) \rangle \, \varepsilon_{y,\mathbf{c}}(t) \, t^k \\ &= \sum_{k=1}^{\infty} \frac{1}{c_k} \langle \varepsilon_1 \mid t_{\mathbf{d}} s_k(xy) \rangle \, \varepsilon_{y,\mathbf{c}}(t) \, t^k \end{aligned}$$

from which the result follows.

If we take $d_n = n!$ then $t_{\mathbf{d}}$ is the ordinary derivative D and Theorem 4.2 gives

$$D \varepsilon_{y,\mathbf{c}}(t) = \left(\sum_{k=0}^{\infty} \frac{1}{c_{k+1}} \langle \varepsilon_1 \mid D s_{k+1}(xy) \rangle \, t^k \right) \varepsilon_{y,\mathbf{c}}(t).$$

Solving this differential equation we have

THEOREM 4.3. *Let $s_n(x)$ be Sheffer for $(\varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$. Then*

$$\varepsilon_{y,\mathbf{c}}(t) = \exp \sum_{k=1}^{\infty} \frac{1}{k c_k} \langle \varepsilon_1 \mid D s_k(xy) \rangle \, t^k.$$

A situation which seems to occur frequently is when two admissible sequences are related by

$$d_n = \frac{c_n}{\langle N \mid u_n(x) \rangle}$$

where $u_n(x)$ is a sequence of polynomials and N is a linear functional for which $\langle N \mid u_n(x) \rangle \neq 0$. In this case

$$\begin{aligned} \varepsilon_{y,\mathbf{d}}(t) &= \sum_{k=0}^{\infty} \frac{y^k}{d_k} t^k \\ &= \sum_{k=0}^{\infty} \frac{\langle N \mid u_k(x) \rangle}{c_k} (yt)^k \end{aligned}$$

$$\left\langle N \,\middle|\, \sum_{k=0}^{\infty} \frac{u_k(x)}{c_k} (yt)^k \right\rangle$$

where the last equality is, in effect, a definition. This leads to the following result.

THEOREM 4.4. *Let* $\mathbf{c}$ *be an admissible sequence,* $u_n(x)$ *the basic sequence for* $(t_{\mathbf{e}}, \mathbf{c}, M)$ *and* N *a linear functional for which* $\langle N \mid u_n(x)\rangle \neq 0$. *Then if*

$$d_n = \frac{c_n}{\langle N \mid u_n(x)\rangle}$$

we have

$$\varepsilon_{y,\mathbf{d}}(t) = \frac{\langle N \mid \varepsilon_{x,\mathbf{c}}(yt)\rangle}{g_{M,\mathbf{e}}(yt)}.$$

Next we consider the problem of determining $D_k(t)$ in the formal equation

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} D_k(t)$$

where $s_n(x)$ is Sheffer for $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$. Clearly $D_k(t)$ exists and in fact

$$D_k(t) = \langle g(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid \varepsilon_{x,d}(t)\rangle$$

where t is a formal variable and the linear functional $g(t_{\mathbf{c}})\, t_{\mathbf{c}}^k$ acts on the coefficients of the powers of t (which are polynomials in x).

Let us suppose that the operators $t_{\mathbf{c}}$ and $t_{\mathbf{d}}$ are related by

$$t_{\mathbf{c}} = u(\sigma_y)\, t_{\mathbf{d}}$$

where $u(\sigma_y)$ is a linear combination of integral powers of σ_y. Since $t_{\mathbf{d}}\sigma_y^n = y^n \sigma_y^n t_{\mathbf{d}}$ we have

$$t_{\mathbf{c}}^k = u(\sigma_y)\, u(y\sigma_y) \cdots u(y^{k-1}\sigma_y)\, t_{\mathbf{d}}^k.$$

Since $t_{\mathbf{d}}^k \varepsilon_{x,\mathbf{d}}(t) = t^k \varepsilon_{x,\mathbf{d}}(t)$ we get

$$\begin{aligned} D_k(t)&\langle g(t_{\mathbf{c}}) \mid t_{\mathbf{c}}^k \varepsilon_{x,\mathbf{d}}(t)\rangle \\ &= t^k \langle g(t_{\mathbf{c}}) \mid u(\sigma_y) \cdots u(y^{k-1}\sigma_y)\, \varepsilon_{x,\mathbf{d}}(t)\rangle. \end{aligned}$$

Now if

$$l(t) = \sum_{k=0}^{\infty} \frac{\langle g(t_{\mathbf{c}}) \mid x^k\rangle}{d_k} t^k$$

then $\langle l(t_{\mathbf{d}}) \mid p(x)\rangle = \langle g(t_{\mathbf{c}}) \mid p(x)\rangle$ and so

$$D_k(t) = t^k \langle l(t_{\mathbf{d}}) \mid u(\sigma_y) \cdots u(y^{k-1}\sigma y)\, \varepsilon_{x,\mathbf{d}}(t)\rangle$$
$$= t^k \langle u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*)\, l(t_{\mathbf{d}}) \mid \varepsilon_{x,\mathbf{d}}(t)\rangle.$$

Finally, since for any series $f(t)$ we have $\langle f(t_{\mathbf{d}}) \mid \varepsilon_{x,\mathbf{d}}(t)\rangle = f(t)$ we get

$$D_k(t) = t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*)\, l(t).$$

We have proved the following result.

THEOREM 4.5. *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be admissible sequences such that*

$$t_{\mathbf{c}} = u(\sigma_y)\, t_{\mathbf{d}}$$

where $u(\sigma_y)$ *is a finite linear combination of integral powers of* σ_y. *Then if* $s_n(x)$ *is Sheffer for* $(g(t_{\mathbf{c}}), t_{\mathbf{c}})$ *we have*

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k}\, t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*)\, l(t)$$

where

$$l(t) = \sum_{k=0}^{\infty} \frac{\langle g(t_{\mathbf{c}}) \mid x^k\rangle}{d_k}\, t^k.$$

COROLLARY 4.6. *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be as in Theorem* 4.5 *and suppose* $s_n(x)$ *is Sheffer for* $(\varepsilon_{z,\mathbf{c}}(t_{\mathbf{c}}), t_{\mathbf{c}})$. *Then*

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k}\, t^k u(\sigma_y^*) \cdots u(y^{k-1}\sigma_y^*)\, \varepsilon_{z,\mathbf{d}}(t).$$

5. THE q-CASE

Let us first set down the basic facts of the q-case. We set

$$c_n = \frac{(1-q)(1-q^2)\cdots((1-q^n)}{(1-q)^n}.$$

Then

$$\frac{c_n}{c_{n-1}} = \frac{1-q^n}{1-q}$$

and

$$\frac{c_n}{c_k c_{n-k}} = \binom{n}{k}_q$$

is the q-binomial coefficient.

The operator t_c is known as the q-derivative and we have

$$t_c x^n = \frac{1-q^n}{1-q} x^{n-1}$$

and so

$$t_c p(x) = \frac{p(x)-p(qx)}{(1-q)\,x}.$$

Thus if $s_n(x)$ is Sheffer for t_c we immediately have the recurrence

$$s_n(x) - s_n(qx) = (1-q^n)\, x s_{n-1}(x). \tag{5.1}$$

The operator t_c is related to σ_q by

$$\sigma_q = 1-(1-q)\, x t_c.$$

The operator ∂_c is given by

$$\partial_c f(t) = \frac{f(t)-f(qt)}{(1-q)\,t} \tag{5.2}$$

and satisfies the Leibnitz formula (see [8] for an umbral proof)

$$\partial_c^n f(t)\, g(t) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-n)} \partial_c^k f(t)\, \partial_c^{n-k} g(q^k t).$$

The q-exponential series

$$\varepsilon_{y,c}(t) = \sum_{k=0}^{\infty} \frac{1}{c_k} (yt)^k$$

satisfies $\partial_c \varepsilon_{y,c}(t) = y\varepsilon_{y,c}(t)$, which using (5.2) is

$$\varepsilon_{y,c}(qt) = (1-(1-q)\, yt)\, \varepsilon_{y,c}(t).$$

We also note that

$$\varepsilon_{y,c}(t) = \varepsilon_{1,c}(yt)$$

$$\partial_c \varepsilon_{y,c}(t)^{-1} = -y\varepsilon_{y,c}(qt)^{-1}$$

$$\partial_c \varepsilon_{y,c}(t)\, \varepsilon_{z,c}(t) = (y+z-(1-q)\, yzt)\, \varepsilon_{y,c}(t)\, \varepsilon_{z,c}(t). \tag{5.3}$$

The q-binomial coefficients have important combinatorial significance [6, 7]. In [7] it is shown that when q is a prime power the number $\binom{n}{k}_q$ is the number of vector subspaces of dimension k of a vector space of dimension n over the field $GF[q]$. Following [6] we set

$$G_n = \sum_{k=0}^{n} \binom{n}{k}_q,$$

$$A_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q.$$

Then G_n is the number of subspaces of an n-dimensional vector space over $GF[q]$.

Now we first observe that

$$\left\langle \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^k}{c_k} \,\middle|\, x^n \right\rangle = \binom{n}{k}_q.$$

From this we obtain some of the simplest properties of $\binom{n}{k}_q$,

$$\begin{aligned}\binom{n}{k}_q &= \left\langle \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^k}{c_k} \,\middle|\, x^n \right\rangle \\ &= \frac{c_{k-1}}{c_k} \left\langle \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^{k-1}}{c_{k-1}} \,\middle|\, t_{\mathbf{c}} x^n \right\rangle \\ &= \frac{c_{k-1}}{c_k} \frac{c_n}{c_{n-1}} \binom{n-1}{k-1}_q \\ &= \frac{1-q^n}{1-q^k} \binom{n-1}{k-1}_q\end{aligned}$$

and

$$\begin{aligned}\binom{n}{k}_q &= \left\langle \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^k}{c_k} \,\middle|\, x^n \right\rangle \\ &= \left\langle \partial_{\mathbf{c}} \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^k}{c_k} \,\middle|\, x^{n-1} \right\rangle \\ &= \left\langle \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \frac{t_{\mathbf{c}}^{k-1}}{c_{k-1}} + \frac{q^k t_{\mathbf{c}}^k}{c_k} \varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}}) \,\middle|\, x^{n-1} \right\rangle \\ &= \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q.\end{aligned}$$

Next, using (5.3) we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}_q y^k z^{n-k} &= \langle \varepsilon_{y,c}(t_c)\,\varepsilon_{z,c}(t_c) \mid x^n\rangle \\ &= \langle (y+z-(1-q)\,yzt)\,\varepsilon_{y,c}(t_c)\,\varepsilon_{z,c}(t_c) \mid x^{n-1}\rangle \\ &= (y+z)\sum_{k=0}^{n-1}\binom{n-1}{k}_q y^k z^{n-1-k} \\ &\quad -(1-q^{n-1})\,yz\sum_{k=0}^{n-2}\binom{n-2}{k}_q y^k z^{n-2-k}.\end{aligned}$$

If we take $y=z=1$ we get the recurrence

$$G_n = 2G_{n-1}-(1-q^{n-1})\,G_{n-2}$$

and if we take $y=-1$ and $z=1$ we get

$$A_n=(1-q^{n-1})\,A_{n-2}.$$

Basic umbral techniques can be used to determine solutions to certain recurrence relations. For example, in [2] Al-Salam determines those sequences $s_n(x)$, Sheffer for t_c, which are orthogonal, that is, which satisfy the recurrence

$$s_{n+1}(x)=(x-b_n)\,s_n(x)-d_n s_{n-1}(x), \tag{5.4}$$

$s_0(x)=1$. After determining b_n and d_n, Al-Salam simply states the generating function of the solution. Let us give a complete, yet simple, umbral solution. Suppose $s_n(x)$ satisfies (5.4) and is Sheffer for $(g(t_c), t_c)$. Applying the linear functional $g(t_c)\,t_c^k$ to (5.4) gives

$$\begin{aligned}\langle g(t_c)\,t_c^k \mid s_{n+1}(x)\rangle &= \langle g(t_c)\,t_c^k \mid (x-b_n)\,s_n(x)\rangle \\ &\quad -\langle g(t_c)\,t_c^k \mid d_n s_{n-}(x)\rangle.\end{aligned} \tag{5.5}$$

Then we observe that

$$\langle g(t_c)\,t_c^k \mid s_{n+1}(x)\rangle = c_k\delta_{k,n+1} = \left\langle \frac{c_k}{c_{k-1}}\,g(t_c)\,t_c^{k-1} \mid s_n(x)\right\rangle,$$

$$\langle g(t_c)\,t_c^k \mid xs_n(x)\rangle = \langle \partial_c\, g(t_c)\,t_c^k \mid s_n(x)\rangle$$

and

$$\langle g(t_c)\,t_c^k \mid s_{n-1}(x)\rangle = c_k\delta_{k,n-1} = \left\langle \frac{c_k}{c_{k+1}}\,g(t_c)\,t_c^{k+1} \mid s_n(x)\right\rangle.$$

Thus from (5.5) we get

$$\frac{c_k}{c_{k-1}} g(t)\, t^{k-1} = \partial_c g(t)\, t^k - b_k g(t)\, t^k - \frac{c_k}{c_{k+1}} d_{k+1}\, g(t)\, t^{k+1}.$$

Using Leibniz' formula $\partial_c g(t)\, t^k = (c_k/c_{k-1})\, g(t)\, t^{k-1} + q^k t^k \partial_c g(t)$, and rearranging gives

$$\partial_c g(t) = q^{-k}\left(b_k + \frac{c_k}{c_{k+1}} d_{k+1}\, t\right) g(t),$$

this for all $k \geqslant 0$. Therefore

$$b_k = q^k b_0$$

$$d_{k+1} = \frac{c_{k+1}}{c_k} q^k d_1$$

and

$$\partial_c g(t) = (b_0 + d_1 t)\, g(t).$$

Recalling Eq. (5.3) we see that

$$g(t) = \varepsilon_{y,c}(t)\, \varepsilon_{z,c}(t)$$

where $y + z = b_0$ and $(1-q)\, yz = d_1$. The generating function for $s_n(x)$ is thus $\varepsilon_{x,c}(t)/\varepsilon_{y,c}(t)\, \varepsilon_{z,c}(t)$.

In [2] Al-Salam also shows that those sequences Sheffer for t_c and orthogonal on the unit circle are characterized by satisfying the recurrence

$$s_{n+1}(x) = (x + \gamma + \beta q^n)\, s_n(x) - \gamma(1-q^n) x s_{n-1}(x). \tag{5.6}$$

An umbral solution to this recurrence is as follows. Using (5.1) in (5.6) we get

$$s_{n+1}(x) = (x + \beta q^n)\, s_n(x) + \gamma s_n(qx).$$

If $s_n(x)$ is Sheffer for $(g(t_c), t_c)$ then applying the linear functional $g(t_c)$ gives

$$0 = \langle \partial_c g(t_c) \mid s_n(x) \rangle + \beta q^n \langle g(t_c) \mid s_n(x) \rangle + \gamma \langle g(qt_c) \mid s_n(x) \rangle.$$

Now $q^n \langle g(t_c) \mid s_n(x) \rangle = q^n c_n \delta_{n,0} = c_n \delta_{n,0} = \langle g(t_c) \mid s_n(x) \rangle$ and so, as before,

$$\partial_c g(t) = -\beta q(t) - \gamma g(qt).$$

Using (5.2) we get

$$\frac{g(qt)}{g(t)}=\frac{1+(1-q)\,\beta t}{1+(1-q)\,\gamma t}.$$

Finally, recalling that $\varepsilon_{y,\mathbf{c}}(qt)/\varepsilon_{y,\mathbf{c}}(t)=1-(1-q)\,yt$ we have

$$g(t)=\frac{\varepsilon_{-\beta,\mathbf{c}}(t)}{\varepsilon_{-\gamma,\mathbf{c}}(t)}$$

and the generating function for $s_n(x)$ is $\varepsilon_{-\gamma,\mathbf{c}}(t)\,\varepsilon_{x,\mathbf{c}}(t)/\varepsilon_{-\beta,\mathbf{c}}(t)$.

Let us return to some examples of Sheffer sequences.

1. We write $s_n(x)=[x]_{y,n}$ for the Sheffer sequence for $(\varepsilon_{y,\mathbf{c}}(t_\mathbf{c}),\, t_\mathbf{c})$. To determine $[x]_{y,n}$ we first observe that $[y]_{y,n+1}=\langle \varepsilon_{y,\mathbf{c}}(t_\mathbf{c}) \mid [x]_{y,n+1}\rangle=0$ and so we may set

$$[x]_{y,n+1}=(x-y)\,r_n(x).$$

Now Leibniz' rule gives

$$\begin{aligned} c_{k+1}\delta_{n+1,k+1} &= \langle \varepsilon_{y,\mathbf{c}}(t_\mathbf{c})\, t_\mathbf{c}^{k+1} \mid [x]_{y,n+1}\rangle \\ &= \langle \varepsilon_{y,\mathbf{c}}(t_\mathbf{c})\, t_\mathbf{c}^{k+1} \mid (x-y)\, r_n(x)\rangle \\ &= \langle (\partial_\mathbf{c}-y)\, \varepsilon_{y,\mathbf{c}}(t_\mathbf{c})\, t_\mathbf{c}^{k+1} \mid r_n(x)\rangle \\ &= \frac{c_{k+1}}{c_k}\, \langle \varepsilon_{y,\mathbf{c}}(qt_\mathbf{c})\, t_\mathbf{c}^{k} \mid r_n(x)\rangle \end{aligned}$$

and since $\varepsilon_{y,\mathbf{c}}(qt)=\varepsilon_{qy,\mathbf{c}}(t)$ we get

$$\langle \varepsilon_{qy,\mathbf{c}}(t_\mathbf{c})\, t_\mathbf{c}^{k} \mid r_n(x)\rangle = c_k\delta_{n,k}.$$

Thus $r_n(x)=[x]_{qy,n}$ and so

$$[x]_{y,n+1}=(x-y)[x]_{qy,n}$$

leading to

$$[x]_{y,n}=(x-y)(x-qy)\cdots(x-q^{n-1}y).$$

The generating function for $[x]_{y,n}$ is

$$\sum_{k=0}^{\infty}\frac{[x]_{y,k}}{c_k}\,t^k=\frac{\varepsilon_{x,\mathbf{c}}(t)}{\varepsilon_{y,\mathbf{c}}(t)}.$$

Setting $x=0$ and noticing that

$$[0]_{y,k}=(-y)^k q^{\binom{k}{2}}$$

we get

$$\frac{1}{\varepsilon_{y,\mathbf{c}}(t)}=\sum_{k=0}^{\infty}\frac{(-y)^k}{c_k}q^{\binom{k}{2}}t^k. \tag{5.7}$$

Equation (5.1) is the recurrence

$$[x]_{y,n}-[qx]_{y,n}=(1-q^n)\,x[x]_{y,n-1}.$$

Theorem 2.5 becomes

$$[xy]_{z,n}=\sum_{k=0}^{n}\binom{n}{k}_q [x]_{z,k}\,y^k[yz]_{z,n-k}.$$

The exponential $\varepsilon_{y,\mathbf{c}}(t)$ is related to infinite products. With regard to Theorem 4.3, setting $y=1$, we get

$$\begin{aligned}\langle \varepsilon_1 \mid D[x]_k\rangle &= \langle \varepsilon_1 \mid D(x-1)\cdots(x-q^{k-1})\rangle\\ &=(1-q)\cdots(1-q^{k-1})\\ &=\frac{(1-q)^k}{1-q^k}c_k\end{aligned}$$

and so, as Andrews [3] has observed,

$$\begin{aligned}\varepsilon_{1,\mathbf{c}}(t) &= \exp\sum_{k=1}^{\infty}\frac{1}{1-q^k}\frac{[(1-q)\,t]^k}{k}\\ &=\exp\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}q^{jk}\frac{[(1-q)\,t]^k}{k}\\ &=\exp\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\frac{[q^j(1-q)\,t]^k}{k}\\ &=\exp\sum_{j=0}^{\infty}\log(1-(1-q)\,q^j\,t)^{-1}\\ &=\prod_{j=0}^{\infty}\frac{1}{1-(1-q)\,q^jt}.\end{aligned}$$

Thus

$$\varepsilon_{1,\mathbf{c}}(t/(1-q))=\prod_{j=0}^{\infty}\frac{1}{1-q^jt}.$$

The above calculations can be justified on various grounds, for example, if $|q| < 1$ and $|t| < 1/(1-q)$.

Combining this with the generating function for $[x]_{y,n}$ we have

$$\prod_{j=0}^{\infty} \frac{1-(1-q)\,q^j y^t}{1-(1-q)\,q^j x^t} = \sum_{k=0}^{\infty} \frac{[x]_{y,k}}{c_k} t^k.$$

Replacing t by $t/(1-q)$ and using the notation of basic hypergeometric series, $c_k(1-q)^k = (q;q)_k$ and $[x]_{y,k} = x^k(y/x;q)_k$, this becomes Heine's theorem [11, p. 92]

$$\prod_{j=0}^{\infty} \frac{1-q^j y^t}{1-q^j z^t} = \sum_{k=0}^{\infty} \frac{(y/z;q)_k}{(q;q)_k} (xt)^k$$
$$= {}_1\phi_0[y/x;q;xt].$$

The Expansion Theorem gives

$$\begin{aligned}[x]_{y,n} &= \sum_{k=0}^{n} \frac{\langle t_c^k \mid [x]_{y,n}\rangle}{c_k} x^k \\ &= \sum_{k=0}^{n} \binom{n}{k}_q \langle \varepsilon_{0,c}(t_c) \mid [x]_{y,n-k}\rangle\, x^k \\ &= \sum_{k=0}^{n} \binom{n}{k}_q (-y)^{n-k} q^{\binom{n-k}{2}} x^k\end{aligned}$$

and

$$\begin{aligned}x^n &= \sum_{k=0}^{n} \frac{\langle \varepsilon_{y,c}(t_c)\, t_c^k \mid x^n\rangle}{c_k} [x]_{y,k} \\ &= \sum_{k=0}^{n} \binom{n}{k}_q y^{n-k} [x]_{y,k}.\end{aligned}$$

The Expansion Theorem also gives

$$x[x]_{y,n} = \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,c}(t_c)\, t_c^k \mid x[x]_{y,n}\rangle}{c_k} [x]_{y,k}.$$

But

$$\begin{aligned}\langle \varepsilon_{y,c}(t_c)\, t_c^k \mid x[x]_{y,n}\rangle &= \langle \partial_c \varepsilon_{y,c}(t_c)\, t_c^k \mid [x]_{y,n}\rangle \\ &= \Big\langle \frac{c_k}{c_{k-1}} \varepsilon_{y,c}(t_c)\, t_c^{k-1} \\ &\qquad + (qt_c)^k\, y\varepsilon_{y,c}(t_c) \mid [x]_{y,n} \Big\rangle \\ &= c_k \delta_{n,k-1} + q^k y c_k \delta_{k,n}\end{aligned}$$

and so

$$x[x]_{y,n}=[x]_{y,n+1}+q^n y[x]_{y,n}.$$

In the Expansion Theorem for invariant operators (Theorem 3.2) let us put $p_n(x)=[x]_{1,n}=[x]_n$. Then $L=\varepsilon_1$ and

$$\tau=\sum_{k=0}^{\infty}\frac{1}{c_k}\langle\varepsilon_1\mid\tau[x]_k\rangle\, x^k t_{\mathbf{c}}^k.$$

In particular, if $\tau=\sigma_q^m$ then for $k\geqslant m+1$ we have $[q^m]_k=0$ and for $k\leqslant m$ we have

$$\begin{aligned}[q^m]_k&=(q^m-1)\cdots(q^m-q^{k-1})\\&=(q-1)^k\, q^{\binom{k}{2}}\frac{c_m}{c_{m-k}}\\&=\langle(q-1)^k\, q^{\binom{k}{2}}\varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k\mid x^m\rangle.\end{aligned}$$

Thus

$$\sigma_q^m=\sum_{k=0}^{m}\frac{(q-1)^k}{c_k}\, q^{\binom{k}{2}}\langle\varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k\mid x^m\rangle\, x^k t_{\mathbf{c}}^k.$$

For any polynomial $p(x)$ we get

$$p(\sigma_q)=\sum_{k=0}^{\infty}\frac{(q-1)^k}{c_k}\, q^{\binom{k}{2}}\langle\varepsilon_{1,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k\mid p(x)\rangle\, x^k t_{\mathbf{c}}^k.$$

In case $p(x)=[x]_n$ we obtain

$$[\sigma_q]_n=(q-1)^n\, q^{\binom{n}{2}}x^n t_{\mathbf{c}}^n. \tag{5.8}$$

We shall use this formula later.

2. The Sheffer sequence for $(\varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}, t_{\mathbf{c}})$ is

$$\begin{aligned}H_n(x;y)&=\varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})\, x^n\\&=\sum_{k=0}^{n}\binom{n}{k}_q y^{n-k}x^k.\end{aligned}$$

The polynomials $H_n(x;1)$ are known as the q-Hermite polynomials. We have

$$\sum_{k=0}^{\infty}\frac{H_k(x;y)}{c_k}\, t^k=\varepsilon_{y,\mathbf{c}}(t)\,\varepsilon_{x,\mathbf{c}}(t)$$

and for $|q|<1$, $|t|<1/(1-q)$,

$$\sum_{k=0}^{\infty} \frac{H_k(x;y)}{c_k} t^k = \prod_{j=0}^{\infty} \frac{1}{(1-(1-q)\,q^j yt)(1-(1-q)\,q^j xt)}.$$

Equation (5.1) is

$$H_n(x;y) - H_n(qx;y) = (1-q^n)\,xH_{n-1}(x;y).$$

The Expansion Theorem gives

$$xH_n(x;y) = \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k \mid xH_n(x;y)\rangle}{c_k} H_k(x;y).$$

But

$$\begin{aligned}
&< \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}} \mid xH_n(x;y)\rangle \\
&\quad = \langle \partial_{\mathbf{c}} \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k \mid H_n(x;y)\rangle \\
&\quad = \left\langle \frac{c_k}{c_{k-1}} \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^{k-1} - (qt)^k\, y\varepsilon_{y,\mathbf{c}}(qt_{\mathbf{c}})^{-1} \mid H_n(x;y)\right\rangle \\
&\quad = c_k \delta_{k,n+1} - y\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k \mid H_n(qx;y)\rangle
\end{aligned}$$

and so

$$\begin{aligned}
xH_n(x;y) &= \sum_{k=0}^{n+1} \frac{1}{c_k} (c_k \delta_{k,n+1} - y\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k H_n(qx;y)\rangle)\, H_k(x;y) \\
&= H_{n+1}(x;y) - y \sum_{k=0}^{n+1} \frac{\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k \mid H_n(qx;y)\rangle}{c_k} H_k(x;y) \\
&= H_{n+1}(x;y) - yH_n(qx;y).
\end{aligned}$$

Thus

$$H_{n+1}(x;y) = (x + y\sigma_q)\, H_n(x;y).$$

Also

$$\begin{aligned}
x^n &= \sum_{k=0}^{n} \frac{\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1}\, t_{\mathbf{c}}^k \mid x^n\rangle}{c_k} H_k(x;y) \\
&= \sum_{k=0}^{n} \binom{n}{k}_q \langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})^{-1} \mid x^{n-k}\rangle\, H_k(x;y)
\end{aligned}$$

which, using (5.7), gives

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (-y)^{n-k}\, q^{\binom{n-k}{2}} H_k(x;y).$$

The Expansion Theorem also gives

$$H_n(x;z)=\sum_{k=0}^{n}\frac{\langle \varepsilon_{y,\mathbf{c}}(t_{\mathbf{c}})\, t_{\mathbf{c}}^k \mid H_n(x;z)\rangle}{c_k}[x]_{y,k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}_q H_{n-k}(y;z)[x]_{y,k}.$$

Writing out $H_n\,(x;z)$, observing that $H_{n-k}(y;z)=H_{n-k}(z;y)$ and setting $y=1$ gives

$$\sum_{k=0}^{n}\binom{n}{k}_q z^{n-k}x^k=\sum_{k=0}^{n}\binom{n}{k}_q H_{n-k}(z;1)[x]_k.$$

Now in this equation we may replace x by the operator σ_q and since the resulting equation holds for all z, we may replace z by x giving

$$\sum_{k=0}^{n}\binom{n}{k}_q x^{n-k}\sigma_q^k=\sum_{k=0}^{n}\binom{n}{k}_q H_{n-k}(x;1)[\sigma_q]_k.$$

According to (6.8)

$$[\sigma_q]_k\,H_m(x;1)=(q-1)^k\,q^{\binom{k}{2}}\frac{c_m}{c_{m-k}}x^kH_{m-k}(x;1)$$

and so

$$\sum_{k=0}^{n}\binom{n}{k}_q x^{n-k}\sigma_q^kH_m(x;1)$$
$$=\sum_{k=0}^{n}\binom{n}{k}_q\binom{m}{k}_q c_k(q-1)^k\,x^kH_{n-k}(x;1)\,H_{m-k}(x;1).$$

It remains to evaluate the left side of this equation. For $n=1$ we get

$$(x+\sigma_q)\,H_m(x;1)=H_{m+1}(x;1)$$

and for $n=2$, observing that $\sigma_q x=qx\sigma_q$, we have

$$\left[\binom{2}{0}_q x^2+\binom{2}{1}_q x\sigma_q+\binom{2}{2}_q\sigma_q^2\right]H_m(x;1)$$
$$=(x^2+(1+q)\,x\sigma_q+\sigma_q^2)\,H_m(x;1)$$
$$=(x^2+x\sigma_q+\sigma_q x+\sigma_q^2)\,H_m(x;1)$$
$$=(x+\sigma_q)^2\,H_m(x;1)$$
$$=H_{m+2}(x;1).$$

One easily sees, by induction, that

$$\sum_{k=0}^{n} \binom{n}{k}_q x^{n-k}\sigma_q^k = (x+\sigma_q)^n$$

and so we finally have

$$H_{m+n}(x;1) = \sum_{k=0}^{n} \binom{n}{k}_q \binom{m}{k}_q c_k(q-1)^k x^k H_{n-k}(x;1)\, H_{m-k}(x;1),$$

a result first established by Carlitz [5].

6. A Related Umbral Calculus

In this section we shall consider the umbral calculus defined by setting

$$d_n = \frac{c_n}{(-\alpha)^n q^{\binom{n}{2}}} = \frac{c_n}{\langle \varepsilon_0 \mid [x]_{\alpha,n} \rangle}$$

for $\alpha \neq 0$. Then

$$\frac{d_n}{d_{n-1}} = \frac{q^{1-n}}{-\alpha} \frac{c_n}{c_{n-1}}$$

and

$$\frac{d_n}{d_k d_{n-k}} = q^{k(k-n)} \binom{n}{k}_q.$$

The operator $t_{\mathbf{d}}$ satisfies

$$\begin{aligned} t_{\mathbf{d}} x^n &= \frac{q^{1-n}}{-\alpha} t_{\mathbf{c}} x^n \\ &= \frac{1}{-\alpha} \sigma_{q^{-1}} t_{\mathbf{c}} x^n \end{aligned}$$

and so

$$t_{\mathbf{d}} = \frac{1}{-\alpha} \sigma_{q^{-1}} t_{\mathbf{c}}$$

and

$$t_{\mathbf{d}}\, p(x) = \frac{q}{-\alpha} \frac{p(x/q) - p(x)}{x - qx}.$$

If $s_n(x)$ is Sheffer for $t_{\mathbf{d}}$ we have

$$s_n(x/q) - s_n(x) = (q^{-n} - 1)\, x s_{n-1}(x).$$

It is easy to see that

$$\sigma_{q^{-1}} = 1 + (1 - q^{-1})\, \alpha x t_{\mathbf{d}}.$$

The operator $\partial_{\mathbf{d}}$ satisfies a Leibniz-type formula

$$\begin{aligned}\partial_{\mathbf{d}} f(t)\, g(t) &= -\frac{1}{\alpha}\, \sigma^{*}_{q^{-1}} \partial_{\mathbf{c}} f(t)\, g(t)\\ &= f(t/q) \partial_{\mathbf{d}}\, g(t) + q(t)\, \partial_{\mathbf{d}} f(t).\end{aligned}$$

The exponential series $\varepsilon_{y,\mathbf{d}}(t)$ satisfies

$$\varepsilon_{y,\mathbf{d}}(t/q) = (1 + (1 - q^{-1})\, \alpha y t)\, \varepsilon_{y,\mathbf{d}}(t),$$
$$\partial_{\mathbf{d}} \varepsilon_{y,d}(t)^{-1} = -y \varepsilon_{y,\mathbf{d}}(t/q)^{-1}$$

and

$$\partial_{\mathbf{d}} \varepsilon_{y,\mathbf{d}}(t)\, \varepsilon_{z,\mathbf{d}}(t) = (y + z + (1 - q^{-1})\, \alpha y z t)\, \varepsilon_{y,\mathbf{d}}(t)\, \varepsilon_{z,\mathbf{d}}(t).$$

Theorem 4.4 shows that

$$\begin{aligned}\varepsilon_{y,\mathbf{d}}(t) &= \frac{\langle \varepsilon_0 \mid \varepsilon_{x,\mathbf{c}}(yt) \rangle}{\varepsilon_{\alpha,\mathbf{c}}(yt)}\\ &= \frac{1}{\varepsilon_{\alpha,\mathbf{c}}(yt)}.\end{aligned}$$

For $|q| < 1$ and $|t| < 1/(1-q)$ we have

$$\varepsilon_{y,\mathbf{d}}(t) = \prod_{j=0}^{\infty} (1 - (1-q)\, q^j \alpha y t).$$

The connection between **d** and **c** can be seen for $\alpha = -1$ by observing that if q is replaced by q^{-1} in d_n one obtains c_n.

Results similar to those of Section 5 may be obtained here in a completely analogous manner. For example, the numbers

$$G'_n = \sum_{k=0}^{n} q^{k(k-n)} \binom{n}{k}_q$$

satisfy the recurrence

$$G'_n = 2G'_{n-1} - (1 - q^{-(n-1)})\, G'_{n-2}.$$

A solution to the recurrence

$$s_{n+1}(x)=(x-b_n)\,s_n(x)-e_n s_{n-1}(x),$$

$s_0(x)=1$, Sheffer for $t_{\mathbf{d}}$, occurs when

$$b_n=q^{-n}b_0,$$
$$e_{n+1}=-\frac{d_{n+1}}{d_n}q^{-n}\alpha e_1$$

in which case the solution is Sheffer for $(g(t_{\mathbf{d}}), t_{\mathbf{d}})$ where

$$g(t)=\varepsilon_{y,\mathbf{d}}(t)\,\varepsilon_{z,\mathbf{d}}(t)$$

with $y+z=b_0$ and $(q^{-1}-1)\,yz=e_1$.

The main reason for introducing the present umbral calculus is not so much for the above results but rather for results obtained by relating this umbral calculus to the q-umbral calculus (which are to follow).

Let us turn next to Sheffer sequences.

3. The Sheffer sequence for $(\varepsilon_{y,\mathbf{d}}(t_{\mathbf{d}}),\ t_{\mathbf{d}})$ is

$$\begin{aligned}
s_n(x;y)&=\varepsilon_{y,\mathbf{d}}(t_{\mathbf{d}})^{-1}\,x^n\\
&=\varepsilon_{\alpha,\mathbf{c}}(yt_{\mathbf{d}})\,x^n\\
&=\sum_{k=0}^{n}\frac{(\alpha y)^k}{c_k}\,t_{\mathbf{d}}^k x^n\\
&=\sum_{k=0}^{n}\frac{d_n}{c_k d_{n-k}}\,(\alpha y)^k\,x^{n-k}\\
&=\sum_{k=0}^{n}\binom{n}{k}_q(-y)^{n-k}\,q^{\binom{k}{2}-\binom{n}{2}}x^k.
\end{aligned}$$

In a manner similar to that of Example 1 we see that

$$s_n(x;y)=(x-y)(x-q^{-1}y)\cdots(x-q^{-(n-1)}y).$$

The generating function for $s_n(x;y)$ is

$$\begin{aligned}
\sum_{k=0}^{n}\frac{s_k(x;y)}{d_k}\,t^k&=\frac{\varepsilon_{x,\mathbf{d}}(t)}{\varepsilon_{y,\mathbf{d}}(t)}\\
&=\frac{\varepsilon_{\alpha,\mathbf{c}}(yt)}{\varepsilon_{\alpha,\mathbf{c}}(xt)}.
\end{aligned}$$

For $|q|<1$ and $|t|<1/(1-q)$

$$\sum_{k=0}^{n} \frac{s_k(x;y)}{d_k} t^k = \prod_{j=0}^{\infty} \frac{1-(1-q)\,q^j\alpha xt}{1-(1-q)\,q^j\alpha yt}.$$

4. The Sheffer sequence for $(\varepsilon_{y,\mathbf{d}}(t_{\mathbf{d}})^{-1}, t_{\mathbf{d}})$ is

$$\begin{aligned} G_n(x;y) &= \varepsilon_{y,\mathbf{d}}(t_{\mathbf{d}})\, x^n \\ &= \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-n)} y^{n-k} x^k. \end{aligned}$$

These polynomials were introduced by Carlitz [5] for $y=1$. The generating function is

$$\begin{aligned} \sum_{k=0}^{n} \frac{G_k(x;y)}{d_k} t^k &= \varepsilon_{y,\mathbf{d}}(t)\, \varepsilon_{x,\mathbf{d}}(t) \\ &= \frac{1}{\varepsilon_{\alpha,\mathbf{c}}(yt)\, \varepsilon_{\alpha,\mathbf{c}}(xt)} \end{aligned}$$

and for $|q|<1$, $|t|<1/(1-q)$

$$\sum_{k=0}^{n} \frac{G_k(x;y)}{d_k} t^k = \prod_{j=0}^{\infty} (1-(1-q)\,q^j\alpha yt)(1-(1-q)\,q^jxt).$$

As in the q-Hermite case we find that

$$G_{n+1}(x;y) = (x + y\sigma_{q^{-1}})\, G_n(x;y).$$

Now that we have two distinct umbral calculi we may consider Corollary 4.6. Since $t_{\mathbf{c}} = -\alpha\sigma_q t_{\mathbf{d}}$ we have $y=q$ and $u(\sigma_q) = -\alpha\sigma_q$. Thus

$$\begin{aligned} u(\sigma_q^*)\cdots u(q^{k-1}\sigma_q^*)\, \varepsilon_{z,\mathbf{d}}(t) &= (-\alpha)^k q^{\binom{k}{2}} \sigma_q^{*k} \varepsilon_{z,\mathbf{d}}(t) \\ &= (-\alpha)^k q^{\binom{k}{2}} \varepsilon_{z,\mathbf{d}}(q^k t) \end{aligned}$$

and Corollary 4.6 becomes

$$\varepsilon_{x,\mathbf{d}}(t) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{c_k} q^{\binom{k}{2}} [x]_{z,k}\, t^k \varepsilon_{z,\mathbf{d}}(q^k t).$$

Now we divide both sides by $\varepsilon_{z,\mathbf{d}}(t)$ and observe that for $|q|<1$, $|t|<1/(1-q)$,

$$\begin{aligned} \frac{\varepsilon_{z,\mathbf{d}}(q^k t)}{\varepsilon_{z,\mathbf{d}}(t)} &= \prod_{j=0}^{\infty} \frac{1-(1-q)\,q^{j+k}\alpha zt}{1-(1+q)\,q^j\alpha zt} \\ &= \prod_{j=0}^{k-1} \frac{1}{1-(1-q)\,q^j\alpha zt} \end{aligned}$$

and so

$$\prod_{j=0}^{\infty} \frac{1-(1-q)\,q^j\alpha xt}{1-(1-q)\,q^j\alpha zt}$$
$$= \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{c_k} q^{\binom{k}{2}} [x]_{z,k}\, t^k \sum_{j=0}^{k-1} \frac{1}{1-(1-q)\,q^j\alpha zt}.$$

Replacing t by $t/(1-q)$ and setting $\alpha=1$ gives

$$\prod_{j=0}^{\infty} \frac{1-q^jxt}{1-q^jzt} = \sum_{k=0}^{\infty} \frac{(z/x;q)_k}{(z;q)_k} \frac{q^{\binom{k}{2}}}{(q;q)_k} (-xt)^k.$$

7. Another Umbral Calculi

Let us take

$$e_n = \frac{c_n}{[a]_{\alpha,n}} = \frac{c_n}{\langle \varepsilon_a \mid [x]_{\alpha k n} \rangle}$$

where $a \neq 0$ and $a \neq \alpha$. Then

$$\frac{e_n}{e_{n-1}} = \frac{1}{a-\alpha q^{n-1}} \frac{c_n}{c_{n-1}},$$
$$\frac{e_n}{e_k e_{n-k}} = \frac{[a]_{\alpha,k}[a]_{\alpha,n-k}}{[a]_{\alpha,n}} \binom{n}{k}_q.$$

In the notation of basic hypergeometric series $[a]_{\alpha,n} = a^n(\alpha/a;q)_n$ and so

$$e_n = \frac{c_n}{a^n(\alpha/a;q)_n}$$

and

$$\frac{e_n}{e_n e_{n-k}} = \frac{(\alpha/a;q)_k(\alpha/a;q)_{n-k}}{(\alpha/a;q)_n} \binom{n}{k}_q.$$

Observing that

$$(a-\alpha\sigma_q)\, t_{\mathbf{e}} x^n = \frac{e_n}{e_{n-1}} (a-\alpha\sigma^{n-1})\, x^{n-1}$$
$$= t_{\mathbf{c}} x^n$$

we have

$$t_{\mathbf{e}} = \frac{1}{a-\alpha\sigma_q} t_{\mathbf{c}}.$$

According to Theorem 4.4

$$\varepsilon_{y,\mathbf{e}}(t) = \frac{\langle \varepsilon_a \mid \varepsilon_{x,\mathbf{c}}(yt) \rangle}{\varepsilon_{\alpha,\mathbf{c}}(yt)}$$
$$= \frac{\varepsilon_{a,\mathbf{c}}(yt)}{\varepsilon_{\alpha,\mathbf{c}}(yt)}.$$

For $|q| < 1$ and $|t| < 1/(1-q)$

$$\varepsilon_{y,\mathbf{e}}(t) = \prod_{j=0}^{\infty} \frac{1-(1-q)\, q^j \alpha yt}{1-(1-q)\, q^j ayt}.$$

We also remark that

$$\varepsilon_{y,\mathbf{e}}(qt) = \frac{\varepsilon_{ay,\mathbf{c}}(qt)}{\varepsilon_{\alpha y,\mathbf{c}}(qt)}$$
$$= \frac{1-(1-q)\, ayt}{1-(1-q)\, \alpha yt}\, \varepsilon_{y,\mathbf{e}}(t).$$

Let us have more Sheffer sequences.

5. The Sheffer sequence for $(\varepsilon_{y,\mathbf{e}}(t_{\mathbf{e}}), t_{\mathbf{e}})$ is

$$r_n(x;y) = \varepsilon_{y,\mathbf{e}}(t_{\mathbf{e}})^{-1} x^n$$
$$= \frac{\varepsilon_{\alpha,\mathbf{c}}(yt_{\mathbf{e}})}{\varepsilon_{a,\mathbf{c}}(yt_{\mathbf{e}})} x^n$$
$$= \sum_{k=0}^{\infty} \frac{[\alpha]_{a,k}}{c_k}\, y^k t_{\mathbf{e}}^k x^n$$
$$= \sum_{k=0}^{n} \frac{e_n}{c_k e_{n-k}}\, [\alpha]_{a,k}\, y^k x^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k}_q \frac{[\alpha]_{a,n-k}[a]_{\alpha,k}}{[a]_{\alpha,n}}\, y^{n-k} x^k.$$

The generating function is (cf. [3, p. 365, Eq. 9.4])

$$\sum_{k=0}^{\infty} \frac{r_k(x;y)}{e_k} t^k = \frac{\varepsilon_{x,\mathbf{e}}(t)}{\varepsilon_{y,\mathbf{e}}(t)}$$
$$= \frac{\varepsilon_{a,\mathbf{c}}(xt)\, \varepsilon_{\alpha,\mathbf{c}}(yt)}{\varepsilon_{\alpha,\mathbf{c}}(xt)\, \varepsilon_{a,\mathbf{c}}(yt)}.$$

For $|q|<1$, $|t|<1/(1-q)$,

$$\sum_{k=0}^{\infty} \frac{r_k(x; y)}{e_k} t^k = \prod_{j=0}^{\infty} \frac{(1-(1-q)\, q^j ayt)(1-(1-q)\, q^j \alpha xt)}{(1-(1-q)\, q^j yt)(1-(1-q)\, q^j axt)}.$$

Theorem 2.5 is

$$r_n(xy; z) = \sum_{k=0}^{n} \frac{e_n}{e_k e_{n-k}} r_k(x; z)\, y^k r_{n-k}(yz; z).$$

We also have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{r_k(x; y)}{e_k} t^k &= \frac{\varepsilon_{x,\mathbf{c}}(at)\, \varepsilon_{y,\mathbf{c}}(\alpha t)}{\varepsilon_{y,\mathbf{c}}(at)\, \varepsilon_{x,\mathbf{c}}(\alpha t)} \\
&= \sum_{j=0}^{\infty} \frac{[x]_{y,j}}{c_j} (at)^j \sum_{i=0}^{\infty} \frac{[y]_{x,i}}{c_i} (\alpha t)^i \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j}_q \frac{a^j \alpha^{k-j}}{[a]_{\alpha,k}} [x]_{y,j} [y]_{x,k-j} \right) \frac{t^k}{e_k}
\end{aligned}$$

and so

$$r_k(x; y) = \sum_{j=0}^{k} \binom{k}{j}_q \frac{a^j \alpha^{k-j}}{[a]_{\alpha,k}} [x]_{y,j} [y]_{x,k-j}.$$

6. The Sheffer sequence for $(\varepsilon_{y,\mathbf{e}}(t_{\mathbf{e}})^{-1}, t_{\mathbf{e}})$ is

$$\begin{aligned}
u_n(x; y) &= \varepsilon_{y,\mathbf{e}}(t_{\mathbf{e}})\, x^n \\
&= \sum \frac{e_n}{e_k e_{n-k}} y^{n-k} x^k.
\end{aligned}$$

The generating function is

$$\begin{aligned}
\sum \frac{u_k(x; y)}{e_k} t^k &= \varepsilon_{y,\mathbf{e}}(t)\, \varepsilon_{x,\mathbf{e}}(t) \\
&= \frac{\varepsilon_{a,\mathbf{c}}(yt)}{\varepsilon_{\alpha,\mathbf{c}}(yt)} \frac{\varepsilon_{a,\mathbf{c}}(xt)}{\varepsilon_{\alpha,\mathbf{c}}(xt)}.
\end{aligned}$$

Next let us consider Corollary 4.6. Here we have $t_{\mathbf{c}} = (a - \alpha\sigma_q)\, t_{\mathbf{e}}$ and so $u(\sigma_q) = a - \alpha\sigma_q$. Thus

$$u(\sigma_q^*) \cdots u(q^{k-1}\sigma_q^*)\, \varepsilon_{z,\mathbf{e}}(t) = (a - \alpha\sigma_q^*) \cdots (a - \alpha_q^{k-1}\sigma_q^*)\, \varepsilon_{z,\mathbf{e}}(t).$$

For practice we take $k=1$,

$$(a-\alpha\sigma_q^*)\,\varepsilon_{z,\mathbf{e}}(t)a\varepsilon_{z,\mathbf{e}}(t)-\alpha\varepsilon_{z,\mathbf{e}}(qt)$$
$$=\left[a-\alpha\frac{1-(1-q)\,azt}{1-(1-q)\,\alpha zt}\right]\varepsilon_{z,\mathbf{e}}(t)$$
$$=\frac{a-\alpha}{1-(1-q)\,\alpha zt}\,\varepsilon_{z,\mathbf{e}}(t).$$

For $k=2$ we have

$$(a-\alpha\sigma_q^*)(a-\alpha q\sigma_q^*)\,\varepsilon_{z,\mathbf{e}}(t)$$
$$=(a-\alpha q\varepsilon_q^*)\frac{a-\alpha}{1-(1-q)\,\alpha zt}\,\varepsilon_{z,\mathbf{e}}(t)$$
$$=\left[a\frac{a-\alpha}{1-(1-q)\,\alpha zt}\right.$$
$$\left.-\alpha q\frac{a-\alpha}{1-(1-q)\,\alpha zqt}\,\frac{1-(1-q)\,azt}{1-(1-q)\,\alpha zt}\right]\varepsilon_{z,\mathbf{e}}(t)$$
$$=\frac{a-\alpha}{1-(1-q)\,\alpha zt}\,\frac{a-\alpha q}{1-(1-q)\,\alpha zqt}\,\varepsilon_{z,\mathbf{e}}(t).$$

From here it is an easy matter to see (by induction) that

$$(a-\alpha\sigma_q^*)\cdots(a-\alpha q^{k-1}\sigma_q^*)\,\varepsilon_{z,\mathbf{e}}(t)$$
$$=\frac{(a-\alpha)\cdots(a-\alpha q^{k-1})}{(1-(1-q)\,\alpha zt)\cdots(1-(1-q)\,\alpha zq^{k-1}(t)}\,\varepsilon_{z,\mathbf{e}}(t)$$
$$=[a]_{\alpha,k}\,\varepsilon_{z,\mathbf{e}}(t)\prod_{j=0}^{k-1}\frac{1}{1-(1-q)\,\alpha zq^jt}.$$

Thus Corollary 4.6 becomes

$$\varepsilon_{x,\mathbf{e}}(t)=\sum_{k=0}^{\infty}\frac{[a]_{\alpha,k}}{c_k}\,[x]_{z,k}\,t^k\left[\prod_{j=0}^{k-1}\frac{1}{1-(1-q)\,\alpha zq^jt}\right]\varepsilon_{z,\mathbf{e}}(t).$$

Dividing both sides by $\varepsilon_{z,\mathbf{e}}(t)$ and replacing t by $t/(1-q)$ gives

$$\prod_{j=0}^{\infty}\frac{(1-q^jazt)(1-q^j\alpha xt)}{(1-q^j\alpha zt)(1-q^jaxt)}=\sum_{k=0}^{\infty}\frac{(\alpha/a;q)_k(z/x;q)_k}{(\alpha zt;q)_k(q;q)_k}\,(axt)^k$$
$$={}_2\phi_1(\alpha/a;z/x;\alpha zt,q;axt).$$

This is the basic analog of Gauss' theorem [11, p. 97].

8. Some Additional Expansions

In this section we shall very briefly consider some expansions, including Carlitz q-analog of the Lagrange inversion theorem [4]. A more thorough discussion must await a sequel to this paper. Let $c_n = (1-q)\cdots(1-q^n)/(1-q)^n$ be as in Section 5.

Recalling the generating function for $[x]_{y,n}$ we begin by observing that

$$\begin{aligned}
&< \varepsilon_{y,\mathbf{c}}(q^{-n}t_{\mathbf{c}})\, t_{\mathbf{c}}^n \mid \varepsilon_{y,\mathbf{c}}(q^{-k+1}t_{\mathbf{c}})^{-1}\, x^k \rangle \\
&= \frac{c_k}{c_{k-n}} \left\langle \frac{\varepsilon_{y,\mathbf{c}}(q^{-n}t_{\mathbf{c}})}{\varepsilon_{y,\mathbf{c}}(q^{-k+1}t_{\mathbf{c}})} \mid x^{k-n} \right\rangle \\
&= \frac{c_k}{c_{k-n}} \left\langle \sum_{j=0}^{\infty} \frac{[yq^{-n}]_{yq^{-k+1},j}}{c_j} \langle t_{\mathbf{c}}^j \mid x^{k-n} \right\rangle \\
&= \frac{c_k}{c_{k-n}} [yq^{-n}]_{yq^{-k+1},k-n} \\
&= c_n \delta_{n,k}.
\end{aligned} \tag{8.1}$$

From this we obtain the expansion

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_{\mathbf{c}}) \mid \varepsilon_{y,\mathbf{c}}(q^{-k+1}t_{\mathbf{c}})^{-1}\, x^k \rangle}{c_k}\, \varepsilon_{y,\mathbf{c}}(q^{-k}t)\, t^k \tag{8.2}$$

valid for all series $f(t)$, since it holds $f(t) = \varepsilon_{y,\mathbf{c}}(q^{-n}t)\, t^n$ for all $n \geqslant 0$. Variations on this expansion are possible since for any invertible $g(t)$ we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_{\mathbf{c}}) \mid g(t_{\mathbf{c}})\, \varepsilon_{y,\mathbf{c}}(q^{-k+1}t_{\mathbf{c}})^{-1}\, x^k \rangle}{c_k}\, g(t)^{-1}\, \varepsilon_{y,\mathbf{c}}(q^{-k}t)\, t^k.$$

Also, from (5.7) we get

$$\begin{aligned}
\varepsilon_{y,\mathbf{c}}(q^{-k+1}t_{\mathbf{c}})^{-1}\, x^k &= \sum_{j=0}^{\infty} \frac{(-yq^{-k+1})}{c_j}\, q^{\binom{j}{2}} t_{\mathbf{c}}^j x^k \\
&= \sum_{j=0}^{k} \binom{k}{j}_q (-yq^{-k+1})^j\, q^{\binom{j}{2}} x^{k-j}
\end{aligned}$$

and so (8.2) can be written

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{c_k} \left[\sum_{j=0}^{k} \binom{k}{j}_q (-yq^{-k+1})^j\, q^{\binom{j}{2}} \langle f(t_{\mathbf{c}}) \mid x^{k-j} \rangle \right] \varepsilon_{y,\mathbf{c}}(q^{-k}t)\, t^k.$$

As a simple example, if we set $f(t)=\varepsilon_{0,c}(t)=1$ then we get

$$1=\sum_{k=0}^{\infty}\frac{(-y)^k}{c_k}q^{-\binom{k}{2}}\varepsilon_{y,\mathbf{c}}(q^{-k}t)\,t^k.$$

Setting $y=1/(1-q)$, multiplying by $\varepsilon_{y,\mathbf{c}}(t)^{-1}$ and observing that $\varepsilon_{y,\mathbf{c}}(q^{-k}t)/\varepsilon_{y,\mathbf{c}}(t)=1/(1-(1-q)\,tq^{-k})\cdots(1-(1-q)\,tq^{-1})$ we have (after replacing t by $t/(1-q)$),

$$\prod_{j=0}^{\infty}(1-tq^j)=\sum_{k=0}^{\infty}\frac{(-1)^k}{c_k(1-q)^k}q^{-\binom{k}{2}}\frac{t^k}{(1-tq^{-k})\cdots(1-tq^{-1})}.$$

In Slater's [11] notation this is

$$\prod_{j=0}^{\infty}(1-tq^j)=\sum_{k=0}^{\infty}(-1)^k\,q^{-\binom{k}{2}}\frac{t^k}{(q^{-k}t;q)_k(q;q)_k}.$$

Incidentally, in the terminology of [9] the sequence $s_k(x)=\varepsilon_{y,\mathbf{c}}(q^{-k+1}t_\mathbf{c})^{-1}\,x^k$ is known as a decentralized Sheffer sequence. Moreover, Theorem 4.4 of [9] gives a formula which shows that if $\langle\varepsilon_{y,\mathbf{c}}(q^{-n}t_\mathbf{c})\,t_\mathbf{c}^n\mid s_k(x)\rangle=c_n\delta_{n,k}$ then $s_k(x)=\varepsilon_{y,\mathbf{c}}(q^{-k+1}t_\mathbf{c})^{-1}\,x^k$. Thus the sequence $s_k(x)$ was not obtained by sheer guesswork. But since we assume no familiarity with [9] we have given the derivation (8.1) above.

In a similar way we may verify that

$$\langle\varepsilon_{y,\mathbf{c}}(q^n t_\mathbf{c})^{-1}\,t_\mathbf{c}^n\mid\varepsilon_{y,\mathbf{c}}(q^{k-1}t_\mathbf{c})\,x^k\rangle=c_n\,\delta_{n,k}.$$

Thus for any $f(t)$ and invertible $g(t)$,

$$f(t)=\sum_{k=0}^{\infty}\frac{\langle f(t_\mathbf{c})\mid g(t_\mathbf{c})\,\varepsilon_{y,\mathbf{c}}(q^{k-1}t_\mathbf{c})\,x^k\rangle}{c_k}\,g(t)^{-1}\,\varepsilon_{y,\mathbf{c}}(q^k t)^{-1}\,t^k. \tag{8.3}$$

As an example, let us take $g(t)=\varepsilon_{y,\mathbf{c}}(t)^{-1}$. Then

$$\begin{aligned}\frac{\varepsilon_{y,\mathbf{c}}(q^m t)}{\varepsilon_{y,\mathbf{c}}(t)}&=\prod_{j=0}^{m-1}(1-(1-q)\,q^j yt)\\&=((1-q)\,yt;q)_m\end{aligned}$$

and taking $y=1/(1-q)$ gives

$$f(t)=\sum_{k=0}^{\infty}\frac{\langle f(t_\mathbf{c})\mid(t;q)_{k-1}x^k\rangle}{c_k}\frac{t^k}{(t;q)_k}.$$

It is easy to verify that

$$\frac{\varepsilon_{y,c}(q^{k-1}t_c)}{\varepsilon_{y,c}(t_c)}\, x^k = x\, \frac{\varepsilon_{y,c}(q^k t_c)}{\varepsilon_{y,c}(t_c)}\, x^{k-1}.$$

Thus for $g(t) = \varepsilon_{y,c}(t)^{-1}$ Eq. (8.3) is

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t_c) \mid x(t_c; q)_k\, x^{k-1} \rangle}{c_k} \frac{t^k}{(t; q)_k}. \tag{8.4}$$

This is Eq. (1.11) of [4], which is Carlitz' q-analog of a special case of the Lagrange inversion formula. To see this we note that

$$\begin{aligned}\langle f(t_c) \mid x(t_c; q)_k\, x^{k-1} \rangle &= \langle \partial_c f(t_c) \mid (t_c; q)_k x^{k-1} \rangle \\ &= \langle (t_c; q)_k\, \partial_c f(t_c) \mid x^{k-1} \rangle \\ &= \langle \partial_c^{k-1} [(t_c; q)_k\, \partial_c f(t_c)] \mid x^0 \rangle.\end{aligned}$$

In the notation of [4] this is $(1-q)^{-k} \Delta_0^{k-1}[(t)_k \Delta f(t)]$ and since $c_k = (q)_k(1-q)^k$ we get (1.11). Similar considerations would lead to other q-expansions, such as (3.14) of [4].

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