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INVERSE RELATIONS FOR CERTAIN SHEFFER SEQUENCES*

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Abstract. Let $s_n(x)$ $(n = 0, 1, 2, \cdots)$ be a so-called Sheffer sequence of polynomials, and let a_n $(n = 0, 1, 2, \cdots)$ be a sequence of the type $a_n = yn + z$ where y and z are constants. An expansion formula for each polynomial $s_n(x)$ in terms of the sequence $s_n(x + a_n)$ $(n = 0, 1, 2, \cdots)$ is derived, and the formula is illustrated by applications to Laguerre, Hermite, and Gegenbauer polynomials.

1. Statement of main result. In 1939 Sheffer [13] initiated serious study of a class of polynomial sequences which have come to be known as Sheffer sequences. See, for example, [2], [11] and [12], where many additional references are given. These sequences have been characterized in a variety of ways, and we choose here to take as our starting point a generating function characterization that Sheffer himself originally gave. To be precise, a polynomial sequence $s_n(x)$ $(n = 0, 1, 2, \cdots)$ is said to be a *Sheffer sequence* if it is generated by a relation of the form

$$G(t) \exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{1.1}$$

where

$$G(t) = \sum_{n=0}^{\infty} g_n t^n \quad (t_0 \neq 0) \quad \text{and} \quad H(t) = \sum_{n=1}^{\infty} h_n t^n \quad (h_1 \neq 0). \tag{1.2}$$

All of the series here and in what follows are formal power series over the real or complex field.

Associated with any given Sheffer sequence $s_n(x)$ is a polynomial sequence $p_n(x)$ $(n = 0, 1, 2, \cdots)$ of *binomial type* generated by

$$\exp(xH(t)) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}, \tag{1.3}$$

where the $H(t)$ is the same as in (1.1). In view of the additivity property of the exponential function, it is evident from (1.3) that the polynomials $p_n(x)$ satisfy the binomial-type identity

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y), \qquad n = 0, 1, 2, \cdots. \tag{1.4}$$

Note too that it follows from (1.1) and (1.3) that a similar relation,

$$s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y), \qquad n = 0, 1, 2, \cdots, \tag{1.5}$$

relates any Sheffer sequence to the sequence of binomial type associated with it.

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Suppose now that a_n $(n=0, 1, 2, \cdots)$ is a sequence of the form

$$a_n = yn + z,$$

where y and z are constants, independent of x and n, and where $a_n \neq 0$ for any n. Very recently in [4] the first author showed that the sequence $s_n(x+a_n)$ $(n=0, 1, 2, \cdots)$ is itself a Sheffer sequence, and we note here that the expansion

$$s_n(x+a_n) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(a_n) s_k(x), \qquad n = 0, 1, 2, \cdots \tag{1.6}$$

of each polynomial $s_n(x+a_n)$ in terms of the sequence $s_n(x)$ is immediate from (1.5).

In the important special case of Appell sequences [1], occurring when $H(t)=t$ in (1.1) and (1.3) and therefore when $p_n(x)=x^n$ in (1.3), a pair of inverse relations obtained by Gould in [5] can be rewritten in such a fashion as to invert (1.6) and thus expand each polynomial $s_n(x)$ in terms of the sequence $s_n(x+a_n)$. To be precise, if we set $a=z$, $b=y$ and put

$$F(n) = \frac{(-1)^n s_n(x)}{n!\, a_n}, \qquad f(n) = \frac{s_n(x+a_n)}{a_n^{n+1}}$$

in Gould's

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+bk)^n}{n!} f(k), \tag{1.7}$$

$$\frac{(a+bn)^n}{n!} f(n) = \sum_{k=0}^{n} (-1)^k \frac{(a+bn)^{n-k}}{(n-k)!} F(k) \frac{a+bk}{a+bn}, \tag{1.8}$$

we find that (1.8) becomes (1.6) and (1.7) becomes

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{a_n}{a_k} p_{n-k}(-a_k) s_k(x+a_k), \qquad n = 0, 1, 2, \cdots. \tag{1.9}$$

Expansion (1.9) is valid, moreover, when $H(t) = \log(1+t)$, in which case

$$p_n(x) = \binom{x}{n} n!.$$

It is readily obtained by setting $a=-z$, $b=1-y$ and writing

$$F(n) = \frac{s_n(x)}{n!\, a_n}, \qquad f(n) = \frac{(-1)^n s_n(x+a_n)}{\binom{n-a_n}{n} n!\, a_n}$$

in the inverse relations

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{n} f(k), \tag{1.10}$$

$$\binom{a+bn}{n} f(n) = \sum_{k=0}^{n} (-1)^k \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} F(k), \tag{1.11}$$

also derived by Gould in [5]. Here (1.11) and (1.10) become (1.6) and (1.9), respectively.

The above suggests that expansion (1.9) may actually be valid for *any* Sheffer sequence, and our main object is to show that this is in fact the case. Once (1.9) has been established, we also have the following expansion, obtained by letting the z in $a_n = yn + z$ and $a_k = yk + z$ there tend to zero:

$$s_n(x) = \sum_{k=0}^{n} c_k s_k(x + yk), \qquad n = 1, 2, \cdots, \tag{1.12}$$

where

$$c_k = \begin{cases} -n!\, nyh_n, & k = 0, \\ \binom{n}{k} \frac{n}{k} p_{n-k}(-yk), & k = 1, 2, \cdots, n, \end{cases} \tag{1.13}$$

the h_n's being the coefficients in (1.2). To see this, we need to pay special attention to the first ($k = 0$) coefficient,

$$(yn + z)\frac{p_n(-z)}{z},$$

in (1.9) since it is undefined when $z = 0$. According to (1.3), however, $p_n(0) = 0$ and $p'_n(0) = h_n n!$ ($n = 1, 2, \cdots$); and l'Hôpital's rule reveals that

$$\lim_{z \to 0} \frac{p_n(-z)}{z} = -p'_n(0) = -h_n n!.$$

This gives c_0, the remaining coefficients in (1.9) being well defined when $z = 0$.

We shall derive (1.9), our main result, in two different ways. The first (§ 2) is more classical in nature and makes direct use of Lagrange's expansion formula. The second (§ 3) relies on the theory of Sheffer sequences from the more modern point of view of linear operators and linear functionals. That point of view has been intensively developed during the past decade and goes by the name *umbral calculus*.

Finally, in § 4 we illustrate the use of (1.9) in obtaining a variety of expansions, many of them evidently new, involving well-known special functions. We confine our illustrations to Laguerre, Hermite, and Gegenbauer polynomials. An extensive listing of other Sheffer sequences to which our main result can be applied is found, for example, in [2].

2. Derivation I. We begin our first derivation of (1.9) by writing the series

$$S = \sum_{n=0}^{\infty} \frac{s_n(x)}{a_n} \frac{t^n}{n!} \tag{2.1}$$

in the form

$$S = \sum_{n=0}^{\infty} [\exp(a_n H(t))] \frac{s_n(x) \exp(-a_n H(t))}{a_n} \frac{t^n}{n!}. \tag{2.2}$$

We then appeal to Lagrange's expansion formula [7, p. 145],

$$F(t) = F(0) + \sum_{k=1}^{\infty} \left\{ \frac{d^{k-1}}{dt^{k-1}} [F'(t)(f(t))^k] \right\}_{t=0} \frac{1}{k!} \left(\frac{t}{f(t)} \right)^k, \tag{2.3}$$

where $F(t)$ and $f(t)$ have formal Maclaurin series expansions and $f(0) \neq 0$. In that formula we put

$$F(t)=\exp(a_nH(t)) \quad \text{and} \quad f(t)=\exp(yH(t)),$$

and also observe from (1.3), when viewed as a formal Maclaurin series with n replaced by k, that $p_0(x)=1$ and

$$\begin{aligned} p_k(x) &= \left\{\frac{d^k}{dt^k}\exp(xH(t))\right\}_{t=0} \\ &= \left\{\frac{d^{k-1}}{dt^{k-1}}[xH'(t)\exp(xH(t))]\right\}_{t=0}, \qquad k=1,2,\cdots. \end{aligned}$$

Equation (2.3) then becomes

$$\exp(a_nH(t))=\sum_{k=0}^{\infty}\frac{a_n}{a_{n+k}}p_k(a_{n+k})\frac{[t\exp(-yH(t))]^k}{k!};$$

using this to substitute for the factor in square brackets in (2.2), we have

$$S=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}\binom{n+k}{k}p_k(a_{n+k})s_n(x)\right\}\frac{\exp(-a_{n+k}H(t))}{a_{n+k}}\frac{t^{n+k}}{(n+k)!},$$

or

$$S=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}p_k(a_n)s_{n-k}(x)\right\}\frac{\exp(-a_nH(t))}{a_n}\frac{t^n}{n!}. \tag{2.4}$$

Now, in view of (1.5), the factor in braces in (2.4) can be written $s_n(x+a_n)$; and so (2.4) becomes

$$S=\sum_{n=0}^{\infty}\frac{s_n(x+a_n)}{a_n}\frac{t^n}{n!}\exp(-a_nH(t)). \tag{2.5}$$

Replacing the variable of summation n here by k and then observing from (1.3) that

$$\exp(-a_kH(t))=\sum_{n=0}^{\infty}p_n(-a_k)\frac{t^n}{n!},$$

we find that

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{p_n(-a_k)}{a_k}s_k(x+a_k)\frac{t^{n+k}}{(n+k)!},$$

or

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{p_{n-k}(-a_k)}{a_k}s_k(x+a_k)\frac{t^n}{n!}. \tag{2.6}$$

Finally, if we equate coefficients of $t^n/n!$ on the right-hand sides of (2.1) and (2.6), we arrive at (1.9).

3. Derivation II. We preface our second derivation of (1.9) with a summary of relevant results from the umbral calculus. In fact, most of this section is devoted to providing background to these recently developed methods, and our second derivation of (1.9) is actually shorter than the first. No proofs are given here; rather we refer the reader to [11]. For even more recent developments and generalizations of the umbral calculus, see [8], [9] and [10].

Let us start by defining three algebras. The first algebra P is the familiar algebra of polynomials in a single variable x over the real or complex field.

The second algebra P^* is the dual vector space of linear functionals on P endowed with the following product. Let L and M be linear functionals. We denote the action of a linear functional N on a polynomial $p(x)$ by $\langle N|p(x)\rangle$, and define the product LM by

$$\langle LM|x^n\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L|x^k\rangle\langle M|x^{n-k}\rangle.$$

It is easy to verify that P^* is an associative and commutative algebra with identity ε defined by

$$\langle \varepsilon|p(x)\rangle = p(0).$$

We call P^* the *umbral algebra*. A particularly important role is played by the *delta functionals*, namely those functionals L for which $\langle L|1\rangle = 0$ and $\langle L|x\rangle \neq 0$. Among these is the *generator* A defined by $\langle A|p(x)\rangle = p'(0)$, where $p'(x)$ is the derivative of $p(x)$. If a is a constant, the *evaluation functional* ε_a is defined by $\langle \varepsilon_a|p(x)\rangle = p(a)$. Note that $\varepsilon_0 = \varepsilon$ where ε is the identity defined above. Finally, we mention that a suitable topology can be put on P^*, allowing us to consider formal power series in a linear functional. It then holds that for any sequence of constants $a_k (k = 0, 1, 2, \cdots)$ the series $\sum_{k=0}^{\infty} a_k L^k$ converges if L is a delta functional. The umbral algebra becomes, moreover, the algebra of all formal power series in the generator A, or in any delta functional (see Theorem D below).

The third algebra S is the algebra of all linear operators on P, under composition, which commute with the derivative operator; that is, the elements of S are all linear operators T such that

$$TDp(x) = DTp(x)$$

for all $p(x) \in P$. We call S the algebra of *shift-invariant operators*. Again with a suitable topology, one may characterize S as the algebra of all formal power series in D.

Thus both P^* and S are isomorphic to the algebra of formal power series in a single variable, and so to each other. In fact, the map $\mu : P^* \to S$ sending the generator A to the derivative D can be extended to a continuous algebra isomorphism of P^* onto S. In other words, if $L = \sum_{k=0}^{\infty} a_k A^k$, then $\mu(L) = \sum_{k=0}^{\infty} a_k D^k$. A *delta operator* is the image of a delta functional under μ. In terms of formal power series, the adjective "delta" means zero constant term and nonzero linear term. The evaluation functional ε_a in P^* corresponds to the *shift operator*

$$E^a = \mu(\varepsilon_a) : p(x) \to p(x+a)$$

in S.

A basic tool of the umbral calculus is the interplay between P^* and S that is described in the following theorem.

THEOREM A. *Let L and M be linear functionals. Then*

$$\langle LM|p(x)\rangle = \langle L|\mu(M)p(x)\rangle$$

for all $p(x) \in P$.

By a *sequence* of polynomials $p_n(x)$ $(n = 0, 1, 2, \cdots)$, we imply that $\deg p_n(x) = n$. A sequence $p_n(x)$ is of *binomial type* if

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y), \qquad n = 0, 1, 2, \cdots, \tag{3.1}$$

for all x and y. A polynomial sequence $s_n(x)$ $(n = 0, 1, 2, \cdots)$ is a *Sheffer sequence* if there is a sequence $p_n(x)$ of binomial type such that

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y), \qquad n = 0, 1, 2, \cdots, \tag{3.2}$$

for all x and y. The reader will recall that this terminology was used in § 1. Characterizations (3.1) and (3.2) are, in fact, equivalent to the generating function characterizations (1.3) and (1.1), respectively, in that earlier section.

Now the key to the present theory is that sequences of Sheffer type (which includes binomial type) may be characterized by means of the algebras P^* and S.

THEOREM B. *A sequence $p_n(x)$ in P is of binomial type if and only if*

(i) *there exists a delta functional L such that*

$$\langle L^k | p_n(x) \rangle = n!\, \delta_{n,k},$$

or, in operator terms,

(ii) (a) $p_n(0) = \delta_{n,0}$,

(b) *there exists a delta operator $T(= \mu(L))$ such that*

$$Tp_n(x) = np_{n-1}(x), \qquad n = 1, 2, \cdots.$$

The sequence $p_n(x)$ is called the *associated sequence* for L (or T).

THEOREM C. *A sequence $s_n(x)$ in P is a Sheffer sequence if and only if*

(i) *there is an invertible linear functional N (i.e., $\langle N|1\rangle \neq 0$) and a delta functional L such that*

$$\langle NL^k | s_n(x) \rangle = n!\, \delta_{n,k},$$

or

(ii) *there exists an invertible shift-invariant operator T and a sequence $p_n(x)$ of binomial type such that*

$$s_n(x) = Tp_n(x),$$

or

(iii) *there exists a delta operator T such that*

$$Ts_n(x) = ns_{n-1}(x), \qquad n = 1, 2, \cdots.$$

The most useful result for our purposes is, however, the Expansion Theorem:

THEOREM D. *Let L be a delta functional with associated sequence $p_n(x)$. Then if M is any linear functional, we have*

$$M = \sum_{k=0}^{\infty} \frac{\langle M | p_k(x) \rangle}{k!} L^k.$$

In terms of shift-invariant operators, if $T = \mu(L)$ and $S = \mu(M)$, we obtain

$$S = \sum_{k=0}^{\infty} \frac{\langle M | p_k(x) \rangle}{k!} T^k.$$

We require one more result to complete our discussion. If T is a delta operator with associated sequence $p_n(x)$, then, for any constant a, $E^a T$ is also a delta operator. Its associated sequence is given by

$$q_n(x) = xE^{-an}x^{-1}p_n(x) = \frac{x}{x-an} p_n(x-an). \tag{3.3}$$

We turn now to the derivation of expansion (1.9) using the umbral calculus. Actually, it is nothing more than a corollary of the Expansion Theorem. Let $s_n(x)$ be a Sheffer sequence and let T be the delta operator in part (iii) of Theorem C. Suppose that $p_n(x)$ is the associated sequence for T. Then, according to (3.3), the delta operator $E^{-y}T$ has the associated sequence

$$q_n(x) = \frac{x}{x+yn} p_n(x+yn).$$

If we write $a_n = yn + z$, the Expansion Theorem gives

$$E^{-a_n} = \sum_{k=0}^{\infty} \frac{\langle \varepsilon_{-a_n} | q_k(x) \rangle}{k!} (E^{-y}T)^k,$$

which may be written as

$$I = \sum_{k=0}^{\infty} \frac{q_k(-a_n)}{k!} E^{a_{n-k}} T^k.$$

Applying this to the polynomial $s_n(x)$, and noticing that

$$q_k(-a_n) = \frac{-a_n}{-a_n + yk} p_n(-a_n + yk) = \frac{a_n}{a_{n-k}} p_k(-a_{n-k}),$$

and

$$E^{a_{n-k}} T^k s_n(x) = \binom{n}{k} k! E^{a_{n-k}} s_{n-k}(x) = \binom{n}{k} k! s_{n-k}(x + a_{n-k}),$$

we obtain

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{a_n}{a_{n-k}} p_k(-a_{n-k}) s_{n-k}(x + a_{n-k}), \qquad n = 0, 1, 2, \cdots.$$

Replacing k by $n-k$ here finally gives (1.9).

4. Applications to special functions. The sequence of Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+n}{n-k} (-x)^k, \qquad n = 0, 1, 2, \cdots \tag{4.1}$$

generated by

$$(1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} \tag{4.2}$$

is a familiar Sheffer sequence. We follow Rota et al. [11], [12] here and in what follows immediately below, where we let $L_n(x)$ denote the basic Laguerre polynomials ($\alpha = -1$). It should be emphasized that other authors often do not include the $n!$ on the right-hand sides of (4.1) and (4.2) and use $L_n(x)$ for the case $\alpha = 0$.

It follows from (1.9) that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{yn+z}{yk+z} L_{n-k}(-yk-z) L_k^{(\alpha)}(x+yk+z), \qquad n = 1, 2, \cdots, \tag{4.3}$$

and (1.12)–(1.13), with $h_n = -1$, tells us that the limiting case of (4.3) as $z \to 0$ is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_k L_k^{(\alpha)}(x+yk), \qquad n = 1, 2, \cdots, \tag{4.4}$$

where

$$c_k = \begin{cases} n!\,ny, & k = 0, \\ \binom{n}{k}\frac{n}{k}L_{n-k}(-yk), & k = 1, 2, \cdots, n. \end{cases} \tag{4.5}$$

As pointed out in [4], $L_n^{(\alpha)}(x)$ is also a Sheffer sequence in the parameter α. For (4.2) can be put into the form

$$(1-t)^{-1}\exp\left(\frac{-xt}{1-t}\right)\exp(-\alpha\log(1-t)) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

Here $s_n(\alpha) = L_n^{(\alpha)}(x)$, and

$$\sum_{n=0}^{\infty} p_n(\alpha)\frac{t^n}{n!} = (1-t)^{-\alpha} = \sum_{n=0}^{\infty} (-1)^n\binom{-\alpha}{n}t^n.$$

Evidently, then,

$$p_n(\alpha) = (-1)^n n!\binom{-\alpha}{n},$$

and (1.9), with $a_n = \beta n + \gamma$, yields

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^{n-k}\frac{n!}{k!}\frac{\beta n+\gamma}{\beta k+\gamma}\binom{\beta k+\gamma}{n-k}L_k^{(\alpha+\beta k+\gamma)}(x), \qquad n = 1, 2, \cdots. \tag{4.6}$$

Note that $h_n = 1/n$, and, according to (1.12)–(1.13), the special case of (4.6) as $\gamma \to 0$ is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_k L_k^{(\alpha+\beta k)}(x), \qquad n = 1, 2, \cdots, \tag{4.7}$$

where

$$c_k = \begin{cases} -n!\,\beta, & k = 0, \\ (-1)^{n-k}\frac{n!}{k!}\frac{n}{k}\binom{\beta k}{n-k}, & k = 1, 2, \cdots, n. \end{cases} \tag{4.8}$$

Expansion (4.6) was obtained earlier in [3], where the limiting case as $\gamma \to 0$ was not noted and where the full generality of Sheffer sequences does not appear. That earlier paper treated only the special case when $H(t) = -\log(1-t)$. As pointed out in [3], (4.6) includes the interesting special case

$$x^n = \sum_{k=0}^{n} (-1)^k\frac{n!}{k!}\frac{\alpha+\beta n+n}{\alpha+\beta k+n}\binom{\alpha+\beta k+n}{n-k}L_k^{(\alpha+\beta k)}(x), \tag{4.9}$$

obtained by putting $\alpha = -n$, then replacing γ by $\alpha + n$, and finally observing from (4.1) that $L_n^{(-n)}(x) = (-x)^n$.

The Hermite polynomials $H_n(x)$ form a Sheffer sequence generated by

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{4.10}$$

and (1.9) is therefore applicable. For brevity, we note here only the following special case, obtained from (1.12)–(1.13) when $h_n = 0$ ($n = 2, 3, \cdots$) and $p_n(x) = (2x)^n$:

$$H_n(x) = \sum_{k=1}^{n} \binom{n}{k}\frac{n}{k}(-2yk)^{n-k}H_k(x+yk), \qquad n = 2, 3, \cdots. \tag{4.11}$$

Finally, except for a factor of $n!$, the sequence of Gegenbauer polynomials $C_n^\lambda(x)$ is of binomial type in the parameter λ since it is generated by

$$(1-2xt+t^2)^{-\lambda}=\sum_{n=0}^{\infty} C_n^\lambda(x)t^n,$$

or

$$\exp[-\lambda \log(1-2xt+t^2)]=\sum_{n=0}^{\infty} C_n^\lambda(x)t^n.$$

Here

$$s_n(\lambda)=p_n(\lambda)=n!\,C_n^\lambda(x),$$

and if we write $a_n=\mu n+\nu$, (1.9) becomes

$$\sum_{k=0}^{n}\frac{1}{\mu k+\nu}C_k^{\lambda+\mu k+\nu}(x)C_{n-k}^{-\mu k-\nu}(x)=\frac{1}{\mu n+\nu}C_n^\lambda(x),\qquad n=0,1,2,\cdots. \tag{4.12}$$

Noticing, moreover, that [6, p. 259]

$$H(t)=-\log(1-2xt+t^2)=\sum_{n=1}^{\infty}\frac{2T_n(x)}{n}t^n,$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind, we find from (1.12)–(1.13) that

$$\sum_{k=1}^{n}\frac{n}{k}C_k^{\lambda+\mu k}(x)C_{n-k}^{-\mu k}(x)=C_n^\lambda(x)+2\mu T_n(x),\qquad n=1,2,\cdots. \tag{4.13}$$

Of particular interest because of their symmetry are the identities

$$\sum_{k=0}^{n}\frac{1}{\mu k+\nu}C_k^{\mu k+\nu}(x)C_{n-k}^{-\mu k-\nu}(x)=0,\qquad n=1,2,\cdots \tag{4.14}$$

and

$$\sum_{k=1}^{n}\frac{n}{k}C_k^{\mu k}(x)C_{n-k}^{-\mu k}(x)=2\mu T_n(x),\qquad n=1,2,\cdots, \tag{4.15}$$

obtained by putting $\lambda=0$ in (4.12) and (4.13), respectively.

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Note added in proof. It has been brought to the authors' attention that the expansion (1.9) is obtained independently and in a somewhat different form by H. Niederhausen in an M.I.T. Technical Report of February 1979 entitled *Sheffer polynomials for computing exact Kolmogorov-Smirnov and Renyi type distributions*, which is to appear in Ann. Statist.

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