Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series A
All Rights Reserved by Academic Press, New York and London

Vol. 63, No. 1, May 1993
Printed in Belgium

The Harmonic Logarithms and the Binomial Formula

STEVEN ROMAN

Department of Mathematics, California State University, Fullerton, California 92634

Communicated by Gian-Carlo Rota

Received March 6, 1991

1. INTRODUCTION

The algebra $\mathscr{P}$ of polynomials in a single variable x provides a simple setting in which to do the "polynomial" calculus. Besides its being an algebra, one of the nicest features of $\mathscr{P}$ is that it is closed under both differentiation and antidifferentiation. That is to say, the derivative of a polynomial is another polynomial, and the antiderivative of a polynomial is another polynomial (provided we ignore the arbitrary constant).

Furthermore, within the algebra $\mathscr{P}$, we have the well-known binomial formula

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z},\ n \geqslant 0. \tag{1}$$

This formula may have been known as early as about 1100 AD, in the works of Omar Khayyam. (Euclid knew the formula for $n=2$ around 300 BC). To be sure, the formula, as we know it today, was stated by Pascal in his *Traité du Triangle Arithmétique* in 1665.

But now suppose we wish to include the negative powers of x in our setting. One possibility is to combine the positive and negative powers of x, by working in the algebra $\mathscr{A}$ of Laurent series of the form

$$\sum_{k=-\infty}^{n} a_k x^k$$

The algebra $\mathscr{A}$ is certainly closed under differentiation, and there is even a binomial formula for *negative* integral powers,

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z},\ n < 0, \tag{2}$$

0097-3165/93 $5.00

due to Newton, which converges for $|x| > |a|$. (The binomial formula also holds for noninteger values of n, but we restrict attention to integer values in this paper.)

The algebra $\mathscr{A}$ does suffer from one drawback, however. It is not closed under antidifferentiation. For there is no Laurent series $f(x)$ with the property that $Df(x) = x^{-1}$. To correct this problem, we must introduce the logarithm $\log x$. As we will see, doing so produces some rather interesting consequences. For it leads us to introduce some previously unstudied functions, which Loeb and Rota have called the *harmonic logarithms.* We also obtain a generalization of the binomial coefficients, and the binomial formulas (1) and (2), which holds for *all* integers n. This generalization is called the *logarithmic binomial formula*, and has the form

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a)\, x^k, \qquad (3)$$

where n is *any* integer, t is any positive integer, and the functions $\lambda_n^{(t)}(x)$ are the harmonic logarithms. When $t = 0$, formula (3) reduces to the traditional binomial formula (1), and when $t = 1$ and $n < 0$, formula (3) becomes formula (2). However, for $t = 1$ and $n \geqslant 0$, or for $t > 1$, we get new formulas. The coefficients

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil$$

are generalizations of the binomial coefficients $\binom{n}{k}$, and are defined for *all* integers n and k. Loeb and Rota refer to these as the *Roman coefficients.*

The first thorough study of the harmonic logarithms was made by Loeb and Rota [3]. Our goal here is to report on some of the more basic aspects of this study.

Before beginning, let us set some notation. The symbol D is used for the derivative with respect to the independent variable. Also, if P is a logical relation that is either true or false, we use the notation (P), due to Iverson, to equal 0 if P is false and 1 if P is true. For example,

$$|x| = x \cdot (-1)^{(x<0)}.$$

2. The Harmonic Logarithms

We begin by letting L be the set of all finite linear combinations, with real coefficients, of terms of the form $x^i(\log x)^j$, where i is any integer, and j is any nonnegative integer. That is, L is the real vector space with basis $\{x^i(\log x)^j \mid i, j \in \mathbb{Z}, j \geqslant 0\}$. Under ordinary multiplication,

$$x^i(\log x)^j \cdot x^u(\log x)^v = x^{i+u}(\log x)^{j+v},$$

L becomes an algebra over the real numbers. Furthermore, the formula

$$Dx^i(\log x)^j = ix^{i-1}(\log x)^j + jx^{i-1}(\log x)^{j-1} \tag{4}$$

shows that L is closed under differentiation, and the formulas

$$\begin{aligned} D^{-1}x^i(\log x)^j &= \frac{1}{i+1}x^{i+1}(\log x)^j - \frac{j}{i+1}D^{-1}x^i(\log x)^{j-1}, \qquad i \neq -1 \\ D^{-1}x^{-1}(\log x)^j &= \frac{1}{j+1}(\log x)^{j+1} \end{aligned} \tag{5}$$

can be used to give an inductive proof showing that L is closed under antidifferentiation. In fact, we can characterize L as follows.

PROPOSITION 2.1. *The algebra L is the smallest algebra that contains both x and x^{-1}, and is closed under differentiation and antidifferentiation.*

Formulas (4) and (5) indicate that, while the basis $\{x^i(\log x)^j\}$ may be suitable for studying the algebraic properties of L, it is not ideal for studying the properties of L that are related to the operators D and D^{-1}. To search for a more suitable basis for L, let us take another look at how the derivative acts on powers of x. If we let

$$\lambda_n^{(0)}(x) = \begin{cases} x^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

then the derivative behaves as follows,

$$D\lambda_n^{(0)}(x) = n\lambda_{n-1}^{(0)}(x),$$

for all integers n. Thinking of the functions $\lambda_n^{(0)}(x)$ as a doubly infinite sequence, as shown in Fig. 1, we see that applying the derivative operator D has the effect of shifting one position to the left, and multiplying by a constant.

Let us introduce the notation

$$\lfloor n \rceil = \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0. \end{cases}$$

⋯	$\lambda_{-3}^{(0)}(x)$	$\lambda_{-2}^{(0)}(x)$	$\lambda_{-1}^{(0)}(x)$	$\lambda_0^{(0)}(x)$	$\lambda_1^{(0)}(x)$	$\lambda_2^{(0)}(x)$	$\lambda_3^{(0)}(x)$	⋯
⋯	0	0	0	1	x	x^2	x^3	⋯

FIGURE 1

Then the functions $\lambda_n^{(0)}(x)$ are uniquely defined by the following properties:

(1) $\lambda_0^{(0)}(x)=1$

(2) $\lambda_n^{(0)}(x)$ has no constant term for $n\neq 0$

(3) $D\lambda_n^{(0)}(x)=\lfloor n\rceil\,\lambda_{n-1}^{(0)}(x)$.

Note that the antiderivative behaves nicely on the functions $\lambda_n^{(0)}(x)$, except when applied to $\lambda_{-1}^{(0)}(x)$. With the understanding that D^{-1} produces no arbitrary constant terms, we can write

$$D^{-1}\lambda_n^{(0)}(x)=\begin{cases}\dfrac{1}{n+1}\lambda_{n+1}^{(0)}(x) & \text{for} \quad n\neq -1\\ 0 & \text{for} \quad n=-1.\end{cases}$$

Following these guidelines, we can introduce a second row of functions $\lambda_n^{(1)}(x)$ into Fig. 1, by starting with $\lambda_0^{(1)}(x)=\log x$, and using conditions similar to conditions (1)–(3). In particular, the conditions

(4) $\lambda_0^{(1)}(x)=\log x$

(5) $\lambda_n^{(1)}(x)$ has no constant term

(6) $D\lambda_n^{(1)}(x)=\lfloor n\rceil\,\lambda_{n-1}^{(1)}(x)$

uniquely define a doubly infinite sequence of functions $\lambda_n^{(1)}(x)$, as shown in Fig. 2.

This figure makes it rather easy to guess the general form of the functions $\lambda_n^{(1)}(x)$.

Proposition 2.2. *The functions $\lambda_n^{(1)}(x)$, uniquely defined by conditions (4)–(6) above, are given by*

$$\lambda_n^{(1)}(x)=\begin{cases}x^n(\log x-h_n) & \textit{for} \quad n\geq 0\\ x^n & \textit{for} \quad n<0,\end{cases}$$

where $h_n=1+1/2+1/3+\cdots+1/n$ for $n>0$ and $h_0=0$.

Row 1:	$\cdots$ $\lambda_{-3}^{(0)}(x)$	$\lambda_{-2}^{(0)}(x)$	$\lambda_{-1}^{(0)}(x)$	$\lambda_0^{(0)}(x)$	$\lambda_1^{(0)}(x)$	$\lambda_2^{(0)}(x)$	$\lambda_3^{(0)}(x)$	$\cdots$
	$\cdots$ 0	0	0	1	x	x^2	x^3	$\cdots$
Row 2:	$\cdots$ $\lambda_{-3}^{(1)}(x)$	$\lambda_{-2}^{(1)}(x)$	$\lambda_{-1}^{(1)}(x)$	$\lambda_0^{(1)}(x)$	$\lambda_1^{(1)}(x)$	$\lambda_2^{(1)}(x)$	$\lambda_3^{(1)}(x)$	$\cdots$
	$\cdots$ x^{-3}	x^{-2}	x^{-1}	$\log x$	$x(\log x-1)$	$x^2(\log x-1-\frac{1}{2})$	$x^3(\log x-1-\frac{1}{2}-\frac{1}{3})$	$\cdots$

Figure 2

Proof. Conditions 4 and 5 are clearly satisfied. As for condition 6, for $n<0$, we have

$$Dx^n = nx^{n-1} = \lfloor n \rceil x^{n-1}.$$

For $n>0$, we have

$$\begin{aligned} Dx^n(\log x - h_n) &= nx^{n-1}(\log x - h_n) + x^{n-1} \\ &= nx^{n-1}\left(\log x - h_n + \frac{1}{n}\right) \\ &= nx^{n-1}(\log x - h_{n-1}) \\ &= \lfloor n \rceil x^{n-1}(\log x - h_{n-1}). \end{aligned}$$

Finally, for $n=0$, we have

$$D(\log x) = x^{-1} = \lfloor 0 \rceil x^{-1}. \quad \blacksquare$$

Note that the behavior of D^{-1} on the functions $\lambda_n^{(1)}(x)$ is even nicer than it is on the functions $\lambda_n^{(0)}(x)$, for assuming no arbitrary constant, we have

$$D^{-1}\lambda_n^{(1)}(x) = \frac{1}{\lfloor n+1 \rceil}\lambda_{n+1}^{(1)}(x).$$

The vector space formed by using the functions $\lambda_n^{(0)}(x)$ and $\lambda_n^{(1)}(x)$ as a basis is closed under differentiation and antidifferentiation, but it is not an algebra. (The functions $(\log x)^t$, for $t>1$, are not in this vector space, for instance.) This prompts us to enlarge our class of functions as follows.

Definition. For all integers n and nonnegative integers t, we define the *harmonic logarithms* $\lambda_n^{(t)}(x)$ of *order* t and *degree* n as the unique functions satisfying the following properties:

(1) $\lambda_0^{(t)}(x) = (\log x)^t$

(2) $\lambda_n^{(t)}(x)$ has no constant term, except that $\lambda_0^{(0)}(x) = 1$

(3) $D\lambda_n^{(t)}(x) = \lfloor n \rceil \lambda_{n-1}^{(t)}(x)$.

This definition allows us (at least in theory) to construct the harmonic logarithms by starting each "row" (that is, the harmonic logarithms of a fixed order) at $\lambda_0^{(t)}(x) = (\log x)^t$. Then we differentiate to get $\lambda_n^{(t)}(x)$ for $n<0$, and antidifferentiate to get $\lambda_n^{(t)}(x)$ for $n>0$. In fact, with the usual understanding about D^{-1}, we can write

$$\lambda_n^{(t)}(x) = a_{n,t} D^{-n}(\log x)^t \tag{6}$$

where the $a_{n,t}$ are constants. In order to determine these constants, we first observe that according to Property (1) of the definition,

$$(\log x)^t = \lambda_0^{(t)}(x) = a_{0,t}(\log x)^t$$

and so $a_{0,t} = 1$. Property (3) tells us that

$$\lfloor n \rceil a_{n-1,t} D^{-(n-1)}(\log x)^t = \lfloor n \rceil \lambda_{n-1}^{(t)}(x) = D\lambda_n^{(t)}(x) = a_{n,t} D^{-(n-1)}(\log x)^t$$

and so

$$a_{n,t} = \lfloor n \rceil a_{n-1,t}.$$

Thus, for $n \geqslant 0$, we have

$$a_{n,t} = n a_{n-1,t} = n(n-1)\, a_{n-2,t} = \cdots = n(n-1) \cdots (1)\, a_{0,t} = n!,$$

and for $n < 0$,

$$\begin{aligned} a_{n,t} &= \frac{a_{n+1,t}}{\lfloor n+1 \rceil} = \frac{a_{n+2,t}}{\lfloor n+1 \rceil \lfloor n+2 \rceil} = \cdots = \frac{a_{n+(-n),t}}{\lfloor n+1 \rceil \lfloor n+2 \rceil \cdots \lfloor n+(-n) \rceil} \\ &= \frac{1}{\lfloor n+1 \rceil \lfloor n+2 \rceil \cdots \lfloor 0 \rceil} = \frac{1}{(n+1)(n+2)\cdots(-1)} \\ &= \frac{1}{(-1)(-2)\cdots(-(-n-1))} = \frac{(-1)^{-n-1}}{(-n-1)!} \end{aligned}$$

This leads us to define, for all integers n,

$$\lfloor n \rceil ! = \begin{cases} n! & \text{for} \quad n \geqslant 0 \\ \dfrac{(-1)^{-n-1}}{(-n-1)!} & \text{for} \quad n < 0. \end{cases}$$

Loeb and Rota have called $\lfloor n \rceil !$ the *Roman factorial.* The notation $\lfloor n \rceil !$ was suggested by Donald Knuth. Thus, we have $a_{n,t} = \lfloor n \rceil !$, and Eq. (6) gives us the following proposition.

PROPOSITION 2.3. *The harmonic logarithms have the form*

$$\lambda_n^{(t)}(x) = \lfloor n \rceil !\, D^{-n}(\log x)^t.$$

Proposition 2.3 can be used to derive an explicit formula for the harmonic logarithms. Since this formula is a bit involved, however, we postpone it until later. First, we want to study the numbers $\lfloor n \rceil !$ and derive the logarithmic binomial formula. We should mention now, however, that the harmonic logarithms $\lambda_n^{(t)}(x)$ form a basis for the algebra L.

3. The numbers $\lfloor n \rceil!$

Some values of $\lfloor n \rceil!$ are given in the following table:

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lfloor n \rceil!$	$-\frac{1}{120}$	$\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{2}$	-1	1	1	1	2	6	24	120

It is well known that $n! = \Gamma(n+1)$, for $n \geqslant 0$, where $\Gamma(z)$ is the Gamma function. The numbers $\lfloor n \rceil!$ can also be expressed in terms of the Gamma function.

Proposition 3.1. *For all integers n,*

$$\lfloor n \rceil! = \begin{cases} \Gamma(n+1) & \text{for } n \geqslant 0 \\ \operatorname*{Res}_{z=n+1} \Gamma(z) & \text{for } n < 0. \end{cases}$$

Proof. The case $n \geqslant 0$ is well-known, and we use it to prove the case $n < 0$. Since

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we have, for $n < 0$,

$$\begin{aligned} \operatorname*{Res}_{z=n+1} \Gamma(z) &= \lim_{z \to n+1} (z-n-1)\,\Gamma(z) = \lim_{z \to n+1} (z-n-1) \frac{\pi}{\Gamma(1-z) \sin \pi z} \\ &\frac{\pi}{\Gamma(-n)} \lim_{z \to n+1} \frac{z-n-1}{\sin \pi z} = \frac{\pi}{(-n-1)!} \lim_{z \to n+1} \frac{1}{\pi \cos \pi z} \\ &= \frac{(-1)^{n+1}}{(-n-1)!} = \lfloor n \rceil!. \quad \blacksquare \end{aligned}$$

The next proposition shows that the numbers $\lfloor n \rceil!$ do behave like the ordinary factorials.

Proposition 3.2. *For all integers n,*

(a) $\lfloor n \rceil! = \lfloor n \rceil \lfloor n-1 \rceil!$

(b) $\dfrac{\lfloor n \rceil!}{\lfloor n-k \rceil!} = \lfloor n \rceil \lfloor n-1 \rceil \cdots \lfloor n-k+1 \rceil \qquad$ *for* $k > 0$.

Proof. For (a), if $n>0$, then $\lfloor n \rceil! = n!$, and the result is well-known. For $n=0$, we have $\lfloor 0 \rceil \lfloor 0-1 \rceil! = 1 \cdot 1 = 1 = \lfloor 0 \rceil!$. Finally, if $n<0$, then $n-1<0$, and so

$$\lfloor n-1 \rceil! = \frac{(-1)^{-n}}{(-n)!} = \frac{1}{n}\frac{(-1)^{n-1}}{(-n-1)!} = \frac{1}{n}\lfloor n \rceil! = \frac{1}{\lfloor n \rceil}\lfloor n \rceil!,$$

from which the result follows. Part (b) is proved by induction using part (a). ∎

The proof of the following result is a straightforward application of the definition of $\lfloor n \rceil!$.

PROPOSITION 3.3. *For all integers n,*

$$\text{(a)}\quad \lfloor n \rceil! \lfloor -n \rceil! = (-1)^{n+(n>0)} \lfloor n \rceil$$

$$\text{(b)}\quad \lfloor n \rceil! \lfloor -n-1 \rceil! = (-1)^{n+(n<0)}.$$

Proposition 3.2 can be used to prove the following result about the harmonic logarithms.

PROPOSITION 3.4. *For all integers n and nonnegative integers k,*

$$D^k \lambda_n^{(t)}(x) = \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!} \lambda_{n-k}^{(t)}(x).$$

4. THE COEFFICIENTS $\lfloor {n \atop k} \rceil$

From the definition of $\lfloor n \rceil!$, it is a natural step to make the following definition. For all integers n and k, we let

$$\left\lfloor {n \atop k} \right\rceil = \frac{\lfloor n \rceil!}{\lfloor k \rceil! \lfloor n-k \rceil!}.$$

Loeb and Rota have called the numbers $\lfloor {n \atop k} \rceil$ the *Roman coefficients.* The notation $\lfloor {n \atop k} \rceil$ was also suggested by Knuth, and is read "Roman n choose k." Here are some special values of these coefficients:

$$\left\lfloor {n \atop 0} \right\rceil = \left\lfloor {n \atop n} \right\rceil = 1, \qquad \left\lfloor {n \atop 1} \right\rceil = \left\lfloor {n \atop n-1} \right\rceil = \lfloor n \rceil,$$

$$\left\lfloor {n \atop -1} \right\rceil = \left\lfloor {n \atop n+1} \right\rceil = \frac{1}{\lfloor n+1 \rceil},$$

$$\left\lfloor {0 \atop k} \right\rceil = \frac{(-1)^{k+(k>0)}}{\lfloor k \rceil}.$$

The next proposition shows that these coefficients really do generalize the binomial coefficients.

PROPOSITION 4.1. *Whenever* $n \geqslant k \geqslant 0$, *or* $k \geqslant 0 > n$, *the Roman coefficients agree with the ordinary binomial coefficients, that is,*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \binom{n}{k}.$$

Proof. When $n \geqslant k \geqslant 0$, we have $\lfloor n \rceil ! = n!$, $\lfloor k \rceil ! = k!$, and $\lfloor n-k \rceil ! = (n-k)!$, in which case the result follows directly from the definition. For $k \geqslant 0 > n$, we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \, \lfloor n-k \rceil !} = \frac{1}{k!} \lfloor n \rceil \lfloor n-1 \rceil \cdots \lfloor n-k+1 \rceil$$
$$= \frac{1}{k!} (n)(n-1) \cdots (n-k+1) = \binom{n}{k}. \quad \blacksquare$$

As the next proposition shows, several of the algebraic properties of the Roman coefficients are generalizations of properties of the ordinary binomial coefficients.

PROPOSITION 4.2. (a) *For all integers n, k, and r,*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ n-k \end{matrix} \right\rceil.$$

(b) *For all integers n, k, and r,*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ r \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ r \end{matrix} \right\rceil \left\lfloor \begin{matrix} n-r \\ k-r \end{matrix} \right\rceil.$$

(c) (*Pascal's formula*). *For any two distinct, nonzero integers n and k, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil.$$

Proof. Parts (a) and (b) are direct consequences of the definition. As for part (c), the conditions on n and k are equivalent to the statements $\lfloor n \rceil = n$, $\lfloor k \rceil = k$, and $\lfloor n-k \rceil = n-k$. Hence, using Proposition 3.2, we have

$$\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil = \frac{\lfloor n-1 \rceil !}{\lfloor k \rceil ! \, \lfloor n-1-k \rceil !} + \frac{\lfloor n-1 \rceil !}{\lfloor k-1 \rceil ! \, \lfloor n-k \rceil !}$$

$$= \frac{\lfloor n-1 \rceil !}{\lfloor k-1 \rceil ! \, \lfloor n-1-k \rceil !} \left(\frac{1}{\lfloor k \rceil} + \frac{1}{\lfloor n-k \rceil} \right)$$

$$= \frac{\lfloor n-1 \rceil !}{\lfloor k-1 \rceil ! \, \lfloor n-1-k \rceil !} \frac{\lfloor n \rceil}{\lfloor k \rceil \lfloor n-k \rceil}$$

$$= \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \, \lfloor n-k \rceil !} = \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil . \quad \blacksquare$$

Now let us consider some results that do not have analogs for the ordinary binomial coefficients.

Proposition 4.3. *For all integers n and k, we have*

(a) $$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ n \end{matrix} \right\rceil = \frac{(-1)^{n-k+(n>k)}}{\lfloor n-k \rceil}$$

(b) $$\left\lfloor \begin{matrix} -n \\ -k \end{matrix} \right\rceil = (-1)^{n+k+(n>0)+(k>0)} \left\lfloor \begin{matrix} k-1 \\ n-1 \end{matrix} \right\rceil$$

(c) (*Knuth's Rotation/Reflection Law*)

$$(-1)^{k+(k>0)} \left\lfloor \begin{matrix} -n \\ k-1 \end{matrix} \right\rceil = (-1)^{n+(n>0)} \left\lfloor \begin{matrix} -k \\ n-1 \end{matrix} \right\rceil .$$

Proof. To prove part (a), we use part (a) of Proposition 3.3,

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ n \end{matrix} \right\rceil = \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \, \lfloor n-k \rceil !} \frac{\lfloor k \rceil !}{\lfloor n \rceil ! \, \lfloor k-n \rceil !}$$

$$= \frac{1}{\lfloor n-k \rceil ! \, \lfloor k-n \rceil !} = \frac{(-1)^{n-k+(n>k)}}{\lfloor n-k \rceil} .$$

To prove part (b), we use part (b) of Proposition 3.3,

$$\left\lfloor \begin{matrix} -n \\ -k \end{matrix} \right\rceil = \frac{\lfloor -n \rceil !}{\lfloor -k \rceil ! \, \lfloor k-n \rceil !} = \frac{(-1)^{n+(n>0)}}{\lfloor n-1 \rceil !} \frac{\lfloor k-1 \rceil !}{(-1)^{k+(k>0)}} \frac{1}{\lfloor k-n \rceil !}$$

$$= (-1)^{n+k+(n>0)+(k>0)} \frac{\lfloor k-1 \rceil !}{\lfloor n-1 \rceil ! \, \lfloor k-n \rceil !}$$

$$= (-1)^{n+k+(n>0)+(k>0)} \left\lfloor \begin{matrix} k-1 \\ n-1 \end{matrix} \right\rceil .$$

As for part (c), we replace $-k$ by $k-1$ in part (b), to get

$$\left\lfloor \begin{matrix} -n \\ k-1 \end{matrix} \right\rceil = (-1)^{n+k-1+(n>0)+(k-1<0)} \left\lfloor \begin{matrix} -k \\ n-1 \end{matrix} \right\rceil.$$

Using the fact that $(-1)^{-1+(k-1<0)} = (-1)^{(k>0)}$, and rearranging, we get the desired result. ∎

5. The Logarithmic Binomial Formula

Now let us turn to the logarithmic binomial formula. The next proposition can be proved by induction using formulas (4) and (5).

Proposition 5.1. *Each harmonic logarithm $\lambda_n^{(t)}(x)$ is a finite linear combination of terms of the form $x^i(\log x)^j$, where i is any integer and j is any nonnegative integer.*

In view of Proposition 5.1, for any positive real number a, we can expand the function $\lambda_n^{(t)}(x+a)$ in a Taylor series that is valid for $|x|<a$, as follows

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \frac{[D^k \lambda_n^{(t)}(x)]_{x=a}}{k!} x^k = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a)\, x^k$$

Proposition 5.2 (Logarithmic Binomial Theorem). *For all integers n,*

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a)\, x^k \qquad (7)$$

valid for $|x|<a$.

Let us first look at the case $t=0$. Since $\lambda_{n-k}^{(0)}(a) = a^{n-k}$ for $n \geqslant k$, and $\lambda_{n-k}^{(0)}(a) = 0$ for $n<k$, the sum on the right hand side of (7) is actually a finite one. Furthermore, since $\lfloor {n \atop k} \rceil = \binom{n}{k}$ when $n \geqslant k \geqslant 0$, formula (7) is just the classical binomial formula (1).

Next, consider the case $t=1$ and $n<0$. Since

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\log x - h_n) & \text{for } n \geqslant 0, \\ x^n & \text{for } n<0, \end{cases}$$

Eq. (7) becomes, for $n<0$,

$$(x+a)^n = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^{n-k} x^k.$$

Interchanging the roles of x and a, and noting that $\left\lfloor {n \atop k} \right\rceil = \binom{n}{k}$ when $k \geqslant 0 > n$, we get the classical binomial formula (2). (Proposition 5.2 tells us only that this is valid for $x > a$, rather than $|x| > a$.) Thus, we see that the logarithmic binomial formula is indeed a generalization of the classical binomial formulas (1) and (2).

When $n > 0$, we may extend the definition of the harmonic logarithms of order 1 by taking

$$\lambda_n^{(1)}(0) = \lim_{x \to 0+} \lambda_n^{(1)}(x) = 0.$$

When $t = 1$ and $n \geqslant 0$, the piecewise definition of $\lambda_n^{(1)}(x)$ suggests that we split the sum on the right side of (7), to get

$$\begin{aligned} \lambda_n^{(1)}(x+a) &= \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(1)}(a)\, x^k + \sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^{n-k} x^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \lambda_{n-k}^{(1)}(a)\, x^k + a^n \sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left(\frac{x}{a}\right)^k \end{aligned}$$

valid for $|x/a| < 1$, $a > 0$. This form is convenient for determining convergence on the boundary.

LEMMA 5.3. *Let $a > 0$. Consider the series*

$$\sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left(\frac{x}{a}\right)^k.$$

(1) *For $n > 0$, this series converges for all $|x| \leqslant a$.*

(2) *For $n = 0$, this series converges for all $|x| \leqslant a$, except $x = -a$.*

Proof. For $k > n \geqslant 0$, we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \, \lfloor n-k \rceil !} = \frac{n!\,(k-n-1)!}{k!\,(-1)^{k-n-1}} = (-1)^{k-n-1} \frac{n!}{k(k-1)\cdots(k-n)}.$$

Therefore, if $n > 0$, and $|x| \leqslant a$, we have

$$\left| \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left(\frac{x}{a}\right)^k \right| \leqslant \frac{n!}{k(k-1)\cdots(k-n)} \leqslant \frac{n!}{k(k-1)}$$

and so the series converges. If $n = 0$, the terms of the series have the form

$$\frac{(-1)^{k-1}}{k} \left(\frac{x}{a}\right)^k$$

and it is well known that this logarithmic series converges for $|x/a| \leqslant 1$, except at $x/a = -1$, or $x = -a$. (See, for example, [2, p. 213].) ∎

Abel's limit theorem [2, p. 177] now allows us to deduce the following proposition.

PROPOSITION 5.4. *The logarithmic binomial formula of order* 1

$$\lambda_n^{(1)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(1)}(a)\, x^k, \tag{8}$$

with $a > 0$, *is valid.*

(1) *For* $|x| < a$, *when* $n < 0$,

(2) *For* $|x| \leqslant a$, $x \neq -a$, *when* $n = 0$,

(3) *For* $|x| \leqslant a$, *when* $n > 0$, *where* $\lambda_n^{(1)}(0) = 0$.

Since

$$\lambda_n^{(1)}(1) = \begin{cases} -h_n & \text{for} \quad n \geqslant 0 \\ 1 & \text{for} \quad n < 0, \end{cases}$$

taking $a = 1$ in (8) gives

$$\lambda_n^{(1)}(x+1) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(1)}(1)\, x^k.$$

For $n > 0$, this is

$$(x+1)^n \left[\log(x+1) - h_n\right] = \sum_{k=0}^{n} \binom{n}{k} (-h_{n-k})\, x^k + \sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil x^k$$

which is valid for $|x| \leqslant 1$, where the left hand side is 0 for $x = -1$. Rearranging terms, we get the following expansion.

PROPOSITION 5.5. *For all integers* $n > 0$,

$$(x+1)^n \log(x+1) = \sum_{k=0}^{n} \binom{n}{k} (h_n - h_{n-k})\, x^k + \sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil x^k$$

for $|x| \leqslant 1$, *where the left side is equal to* 0 *for* $x = -1$.

Setting $x = -1$ in this formula gives the following beautiful formula.

COROLLARY 5.6. *For all integers $n > 0$,*

$$\sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} n \\ k \end{matrix}\right] = \frac{(-1)^{n+1}}{n}.$$

Proof. Setting $x = -1$ in Proposition 5.5, we obtain

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \left[\begin{matrix} n \\ k \end{matrix}\right] (-1)^k &= -\sum_{k=0}^{n} \binom{n}{k} (h_n - h_{n-k})(-1)^k \\
&= -h_n \sum_{k=0}^{n} \binom{n}{k} (-1)^k + \sum_{k=0}^{n} \binom{n}{k} (-1)^k h_{n-k} \\
&= 0 + (-1)^n \sum_{k=1}^{n} \binom{n}{k} (-1)^k \sum_{i=1}^{k} \frac{1}{i} \\
&= (-1)^n \sum_{i=1}^{n} \frac{1}{i} \sum_{k=i}^{n} \binom{n}{k} (-1)^k \\
&= (-1)^n \sum_{i=1}^{n} \frac{1}{i} (-1)^i \binom{n-1}{i-1} \\
&= \frac{(-1)^n}{n} \sum_{i=1}^{n} \binom{n}{i} (-1)^i \\
&= \frac{(-1)^{n+1}}{n}.
\end{aligned}$$

Since $\sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$, the result follows. ∎

6. AN EXPLICIT FORMULA FOR THE HARMONIC LOGARITHMS

We now turn to the matter of finding an explicit expression for the harmonic logarithms. Although these functions are ideal with regard to differentiation and antidifferentiation, their expression in terms of powers of x and $\log x$ is not so simple. (Although it is elegant.)

With the benefit of hindsight, we set

$$f_n^{(t)}(x) = x^n \sum_{j=0}^{t} (-1)^j (t)_j c_n^{(j)} (\log x)^{t-j},$$

where $(t)_j = t(t-1) \cdots (t-j+1)$, $(t)_0 = 1$, and $c_n^{(j)}$ are undetermined constants. Then we determine the constants $c_n^{(j)}$ so that the functions $f_n^{(t)}(x)$ satisfy the definition of the harmonic logarithms. After some straightforward computations, we are led to the following proposition.

PROPOSITION 6.1. *The harmonic logarithms $\lambda_n^{(t)}(x)$ are given by the formula*

$$\lambda_n^{(t)}(x) = x^n \sum_{j=0}^{t} (-1)^j (t)_j \, c_n^{(j)} (\log x)^{t-j}, \tag{9}$$

where $(t)_j = t(t-1)\cdots(t-j+1)$, $(t)_0 = 1$, *and the constants* $c_n^{(j)}$ *are uniquely determined by the initial conditions*

$$(1)\quad c_n^{(0)} = \begin{cases} 1 & \text{for } n \geqslant 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad \text{and} \qquad (2)\quad c_0^{(j)} = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

and the recurrence relation (*for* $j > 0$)

$$(3)\quad n c_n^{(j)} = c_n^{(j-1)} + \lfloor n \rceil \, c_{n-1}^{(j)}.$$

The numbers $c_n^{(j)}$, defined for all integers n and all nonnegative integers j by conditions (1)–(3), are known as the *harmonic numbers*. As we will see, these numbers have some rather fascinating properties. Figure 3 shows some values of the harmonic numbers.

Note that condition (1) of Proposition 6.1 gives us the 0th row of the matrix in Fig. 3, and condition (2) gives us the 0th column. Then we can use condition (3) to fill in the remaining portion of the matrix. For the right portion, we use condition (3) in its given form. (See Fig. 4a.) For the left portion, we use condition (3) in the form $\lfloor n \rceil \, c_{n-1}^{(j)} = n c_n^{(j)} - c_n^{(j-1)}$. (See Fig. 4b.)

Before discussing the properties of the harmonic numbers, we should settle the matter of showing that the harmonic logarithms form a basis for

n=0 ↓

$$j=0 \longrightarrow \left[\begin{array}{cccccccccccccc}
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{137}{60} & \cdots \\
\cdots & -\frac{137}{60} & -\frac{25}{12} & -\frac{11}{6} & -\frac{3}{2} & -1 & 0 & 0 & 1 & \frac{7}{4} & * & * & * & \cdots \\
\cdots & -\frac{15}{8} & -\frac{35}{24} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 1 & \frac{15}{8} & * & * & * & \cdots \\
\cdots & -\frac{17}{24} & -\frac{5}{12} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 1 & \frac{31}{16} & * & * & * & \cdots \\
\cdots & -\frac{3}{24} & -\frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{63}{32} & * & * & * & \cdots \\
\cdots & -\frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{127}{64} & * & * & * & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]$$

Columns sum to n — Columns approach n

FIG. 3. Values of the harmonic numbers $c_n^{(j)}$.

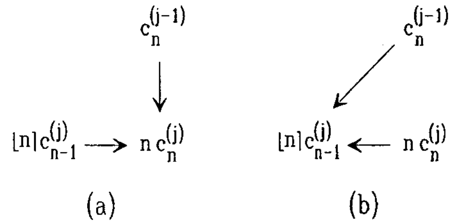

FIGURE 4

the algebra L. If we let $P_{n,t}$ be the sum on the right side of formula (9), then we have

$$\lambda_n^{(t)}(x) = x^n \cdot P_{n,t} = x^n \cdot (c_n^{(0)}(\log x)^t - tc_n^{(1)}(\log x)^{t-1} + \cdots).$$

Now, according to condition (1) of Proposition 6.1, $c_n^{(0)} = 1 \neq 0$ for $n \geqslant 0$, and so, in this case, $P_{n,t}$ has degree t in $\log x$. On the other hand, $c_n^{(0)} = 0$ for $n < 0$. But it is not hard to see from condition (3) in Proposition 6.1 that, for $n < 0$, we have $c_n^{(1)} = -1 \neq 0$. Hence, for $n < 0$, $P_{n,t}$ has degree $t - 1$.

In either case, for any given n, the function $(\log x)^t$ can be written as a finite linear combination of the polynomials $P_{n,j}$. Hence, for each n, the basis functions $x^n(\log x)^t$ for L can be written as a finite linear combination of the harmonic logarithms $\lambda_n^{(t)}(x)$, and so these functions span L. Since the harmonic logarithms are clearly linearly independent, they form a basis for L.

Now let us return to a discussion of the properties of the harmonic numbers. The following initial values of the harmonic numbers follow easily from the definition.

PROPOSITION 6.2. *For the harmonic numbers $c_n^{(j)}$, we have (see Fig. 3)*

(a) (*Column* 0)

$$c_0^{(j)} = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

(b) (*Row* 0)

$$c_n^{(0)} = \begin{cases} 1 & \text{for } n \geqslant 0 \\ 0 & \text{for } n < 0 \end{cases}$$

(c) (*Column* 1)

$$c_1^{(j)} = 1$$

(d) (*Column* -1)

$$c_{-1}^{(j)} = -\delta_{j,1}$$

(e) (*Lower left hand wedge*)

$$c_n^{(j)} = 0 \qquad \text{for } n < 0 \text{ and } j > -n.$$

Let us look more closely at the harmonic numbers $c_n^{(j)}$ of nonnegative degree $n \geqslant 0$. Note the beautiful pattern emerging in part (a).

PROPOSITION 6.3. *For $n > 0$, the harmonic numbers $c_n^{(j)}$ have the following properties*:

(a) $c_n^{(1)} = h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$c_n^{(2)} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)$$

$$\begin{aligned} c_n^{(3)} = 1 &+ \frac{1}{2}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right)\right] \\ &+ \frac{1}{3}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] + \cdots \\ &+ \frac{1}{n}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots \right. \\ &\left. + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)\right]. \end{aligned}$$

In general, for $j > 0$, $\qquad c_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-1)}.$

(b) (*Knuth*)

$$c_n^{(j)} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} i^{-j}$$

(c) *For each $n > 0$, the sequence $c_n^{(j)}$ forms a nondecreasing sequence in j, which is strictly increasing for $n > 1$. Furthermore, we have, for each $n \geqslant 0$,*

$$\lim_{j \to \infty} c_n^{(j)} = n.$$

Proof. For part (a), since $n>0$ and $j>0$, repeated use of condition (3) of Proposition 6.1, for different values of the indices, gives

$$\begin{aligned}
c_n^{(j)} &= \frac{1}{n} c_n^{(j-1)} + c_{n-1}^{(j)} \\
&= \frac{1}{n} c_n^{(j-1)} + \frac{1}{n-1} c_{n-1}^{(j-1)} + c_{n-2}^{(j)} \\
&= \frac{1}{n} c_n^{(j-1)} + \frac{1}{n-1} c_{n-1}^{(j-1)} + \frac{1}{n-2} c_{n-2}^{(j-1)} + c_{n-3}^{(j)} \\
&\vdots \\
&= \frac{1}{n} c_n^{(j-1)} + \frac{1}{n-1} c_{n-1}^{(j-1)} + \frac{1}{n-2} c_{n-2}^{(j-1)} + \cdots + \frac{1}{1} c_1^{(j-1)} + c_0^{(j)}.
\end{aligned}$$

But since $c_0^{(j)}=0$ for $j>0$, the conclusion follows. Part (b) can be proved using Proposition 6.1, but we omit the details.

As for part (c), if $n=0$ or 1, the result follows easily from Proposition 6.2. For each $n>1$, we proceed by induction on j. First, we have $c_n^{(1)}=h_n>1=c_n^{(0)}$. Assuming that $c_n^{(j-1)}>c_n^{(j-2)}$, part (a) gives

$$c_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-1)} > \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-2)} = c_n^{(j-1)}$$

and so $c_n^{(j)}$ is strictly increasing. Furthermore, for a fixed n, $c_n^{(j)}$ is bounded, as can be seen by using part (b):

$$|c_n^{(j)}| \leqslant \sum_{i=1}^{n} \binom{n}{i} i^{-j} \leqslant \sum_{i=1}^{n} \binom{n}{i} = 2^n - 1.$$

Hence, the limit $S_n = \lim_{j\to\infty} c_n^{(j)}$ must exist and be finite. For $n=0$ and $n=1$, we have

$$S_0 = \lim_{j\to\infty} c_0^{(j)} = \lim_{j\to\infty} \delta_{j,0} = 0$$

and

$$S_1 = \lim_{j\to\infty} c_1^{(j)} = \lim_{j\to\infty} 1 = 1.$$

Let us assume that $S_{n-1}=n-1$. Rewriting condition (3) of Proposition 6.1, and taking limits, we have

$$\lim_{j\to\infty} (nc_n^{(j)} - c_n^{(j-1)}) = \lim_{j\to\infty} \lfloor n \rceil c_{n-1}^{(j)} = \lfloor n \rceil S_{n-1} = n(n-1).$$

But since the appropriate limits exist, this can be written

$$\lim_{j\to\infty} nc_n^{(j)} - \lim_{j\to\infty} c_n^{(j-1)} = n(n-1)$$

or

$$nS_n - S_n = n(n-1),$$

from which it follows that $S_n = n$. ∎

To discuss the properties of the harmonic numbers $c_n^{(j)}$ of negative degree $n<0$, we first recall some basic facts about the Stirling numbers $s(n, j)$ of the first kind. These numbers are defined, for all nonnegative integers n and j, by the condition

$$x(x-1)\cdots(x-n+1) = \sum_{j=0}^{n} s(n, j)\, x^j.$$

It can be shown that the Stirling numbers of the first kind are characterized by the following conditions (see [1, p. 214]):

$$\begin{aligned} &s(n, 0) = s(0, j) = 0, \text{ except that } s(0, 0) = 1 \\ &s(n, j) = s(n-1, j-1) + (n-1)\, s(n-1, j). \end{aligned} \tag{10}$$

Now we can state the following proposition.

PROPOSITION 6.4. *For $n<0$, the harmonic numbers have the following properties.*

(a) $c_n^{(1)} = -1$

$$c_n^{(2)} = -h_{-n-1} = -1 - \frac{1}{2} - \cdots - \frac{1}{-n-1}$$

$$c_n^{(3)} = -\frac{1}{2} - \frac{1}{3}\left(1+\frac{1}{2}\right) - \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \cdots - \frac{1}{-n-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{-n-2}\right).$$

In general, for $j>0$,

$$c_n^{(j)} = -\left[\delta_{j,1} + \sum_{i=n+1}^{-1} \frac{1}{i}\, c_i^{(j-1)}\right],$$

where the sum on the right is 0 *if* $n=-1$.

(b) $c_n^{(j)} = (-1)^j \lfloor n \rceil ! \, s(-n, j)$, *where* $s(n, j)$ *are the Stirling numbers of the first kind.*

(c) *For each* $n < 0$, *we have* $c_n^{(j)} = 0$ *for* $j > -n$, *and so only a finite number of the* $c_n^{(j)}$ *are nonzero. Furthermore, we have*

$$\sum_{j=0}^{\infty} c_n^{(j)} = \sum_{j=0}^{-n} c_n^{(j)} = n.$$

(*Contrast this with part* (c) *of Proposition* 6.3.)

Proof. Part (a) can be proved by iteration, in a manner similar to the proof of part (a) of Proposition 6.3. Part (b) can be proved using Proposition 6.1, with the help of Eqs. (10). As for part (c), the first statement follows from Proposition 6.2. For the second statement, we start from the expression, valid for $n < 0$,

$$x(x-1) \cdots (x+n+1) = \sum_{j=0}^{-n} s(-n, j)\, x^j = \frac{1}{\lfloor n \rceil !} \sum_{j=0}^{-n} (-1)^j c_n^{(j)} x^j.$$

Setting $x = -1$, we get

$$(-1)(-2) \cdots (n) = \frac{1}{\lfloor n \rceil !} \sum_{j=0}^{-n} c_n^{(j)}.$$

But by Proposition 3.3, for $n < 0$,

$$(-1)(-2) \cdots (n) \lfloor n \rceil ! = (-1)^n \lfloor -n \rceil ! \, \lfloor n \rceil ! = (-1)^n (-1)^n \lfloor n \rceil = n,$$

from which the result follows. ▮

7. Concluding Remarks

We have merely scratched the surface in the study of the algebra L and its differential operators. For example, the harmonic logarithms $\lambda_n^{(t)}(x)$ have a very special relationship with the derivative operator, spelled out in the definition of these functions. Loeb and Rota show that there are other, at least formal, functions that bear an analogous relationship to other operators, such as the forward difference operator Δ defined by $\Delta p(x) = p(x+1) - p(x)$. The functions associated with the operator Δ are denoted by $(x)_n^{(t)}$ and called the *logarithmic lower factorial functions.* In general, the sequences $p_n^{(t)}(x)$ associated with various operators can be characterized in several ways, for example as sequences of *logarithmic binomial type*, satisfying the identity

$$p_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(a)\, p_{n-k}^{(t)}(x).$$

We hope that the results of this paper justify speaking of the Roman coefficients as a worthy generalization of the binomial coefficients. (This is not to suggest that there may not be other worthy generalizations.) It would be a further confirmation of this fact to discover a nice combinatorial, or probabilistic, interpretation of the Roman coefficients, which, as far as I know, has not yet been accomplished.

References

1. L. Comtet, "Advanced Combinatorics," Reidel, Dordrecht, 1974.
2. K. Knopp, "Theory and Application of Infinite Series," Dover, New York, 1990.
3. D. Loeb and G.-C. Rota, Formal power series of logarithmic type, *Adv. Math.* **75** (1989), 1–118.
4. S. Roman, "The Umbral Calculus," Academic Press, New York, 1984.
5. S. Roman, The algebra of formal series, *Adv. Math.* **31** (1979), 309–329.
6. S. Roman, The algebra of formal series. II. Sheffer sequences, *J. Math. Anal. Appl.* **74** (1980), 120–143.
7. S. Roman and G.-C. Rota, The umbral calculus, *Adv. Math.* **27** (1978), 95–188.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium