Abstract Algebra

A Comprehensive Introduction

Volume 3: Ring Theory
Version 2.0.0

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To Donna

Preface to the Series

This is a multi-volume series on abstract algebra, designed for the serious undergraduate or beginning graduate student. Dependencies among the volumes in this series are as follows.

1) Volume 1: *Linear Algebra* can be read independently of the other volumes.
2) Volume 2: *Group Theory* can be read independently of the other volumes.
3) Volume 3: *Ring Theory* does reference material on both linear algebra and group theory, but any reader with a modest acquaintance with these subjects should be able to read this volume.
4) Volume 4: *Field Theory* does reference material on linear algebra, group theory and ring theory, but any reader with a modest acquaintance with these subjects should be able to read this volume.
5) Volume 5: *Order and Lattices* can be read independently of the other volumes.

As the title suggests, this series is a *comprehensive introduction* to abstract algebra, where by *introduction*, I mean that the book starts more or less at the beginning, assuming no prior knowledge of abstract algebra. The only prerequisites for this series are an understanding of basic mathematical tools as found in a typical "transition course" and a solid understanding of elementary linear algebra as taught in a "relatively serious" lower division course. Certainly, some experience beyond these prerequisites will be of significant benefit in absorbing the material presented in this series.

Realizing that this may be a first exposure to rigorous abstract mathematics for some readers, I have tried to write the discussions and the proofs with an eye towards developing sound mathematical thinking patterns on the part of the reader. Proofs that appear in textbooks often do not show the *thought pattern* behind the discovery of the proof, often preferring elegance or conciseness over insight or motivation. However, showing a bit of the motivation behind a proof is in my opinion extremely valuable for students. Of course, one can easily get carried away with this idea, so I have tried to temper it with an eye towards not slowing down the flow of the books.

By *comprehensive*, I mean that the books in this series include a somewhat wider array of topics (some marked as optional or placed in topics chapters) than is often seen in elementary treatments, so that hopefully *all* readers will find something new and of interest in these books. My goal is for all readers to leave the series with a thorough grounding in the fundamentals of abstract algebra, well prepared to attack more advanced treatments if desired.

It seems that the current trend in mathematical education is to motivate abstract concepts by introducing *applications* as quickly as possible, in an effort to satisfy those students whose overriding question is: Of what use is this material?

While I certainly respect the views of those whose main concern is whether or not the subject matter at hand has applications to the "real" world, I have chosen to take a more abstract approach to the subject of abstract algebra in this series. I am a pure mathematician and appreciate mathematics as an *art form*, as well as the cornerstone of all science and technology. Merriam-Webster defines art as follows:

> something that is created with imagination and skill and that is beautiful or that expresses important ideas or feelings

What could possibly fit this description more accurately than mathematics and in particular, abstract algebra?

Preface to this Volume

This volume of the series is devoted to the elementary theory of rings. For this volume, readers are expected to have a grasp of elementary linear algebra and elementary group theory.

Since this is the third volume in the abstract algebra sequence, I feel that I can rely a bit more on the reader's mathematical maturity and accordingly, I will be a bit less generous with details as I was in the previous courses, relying on you to sort out some of the details.

But I still wish to offer my advice on how to read this volume. I were to give my readers only one single piece of advice, it would be to *constantly question*. If I were to give my readers a second piece of advice, it would be to practice, practice, practice. That is why the book has exercises. The more exercises you attempt, the more easily you will absorb the material. It's that simple (sort of). Along with trying the exercises, whenever you read the statement of a theorem, you should pause a few moments to see if you can construct a proof before reading my proof. Also, any theorem stated without proof is a tacit invitation for you to provide a proof.

Differences and Similarities

If you have followed the first two volumes of this series, then you are about to embark on a study of your third major algebraic structure, namely, rings. In the Introduction to the group-theory course, we discussed the common themes that run through much of abstract algebra and we did so in the context of a widget. Some of these common themes are

- The definition and simple consequences
- Subwidgets
- Morphisms of widgets
- Quotient widgets

and we can now add to this list some items that would have made no sense before studying at least one algebraic structure,

- Isomorphism theorems
- The correspondence theorem

As you will see, we will begin this course on ring theory by studying each of these themes as they apply to rings. While much of this material will seem quite familiar, of course there are specifics that apply to rings alone. Perhaps the biggest such difference is that of the substructure required in order to form a quotient widget. As we have seen, one can factor a vector space by *any* of its subspaces to obtain a quotient space. This is not the case for groups, where we must factor by *normal* subgroups in order to obtain a quotient *group*. As to rings, things are a bit more complicated. Since we require that all subrings of a ring possess a multiplicative identity, it turns out that we cannot factor a ring by any subring other than the trivial subring or the entire ring and expect top get a quotient ring, so we need a brand new concept here, namely, that of an *ideal*.

Once the customary first steps described above are dispatched, further study of linear algebra, group theory, ring theory and field theory, take a decidedly different direction. Specifically, in the course on linear algebra, we devoted a great deal of time and effort in the study of *similarity of matrices*, a concept which has no immediate or obvious parallel in group, ring or field theory. This culminated in the construction of a complete invariant for similarity, namely, the multiset of elementary divisors (or invariant factors) of a matrix.

On the other hand, in the group theory course, we spent our time analyzing the structure of finite groups using group actions, culminating in the Sylow theorems. This enabled us to determine the structure of all groups of order 15 or less.

As to ring theory, it seems (arguably) that the theory of the structure of rings is less accessible *at the introductory level* than that of the theory of the structure of finite groups. Accordingly, after the customary first steps for ring theory, we will turn—as is customary—to a study of a variety of special types of rings, specifically, *integral domains*, *principal ideal domains*, *unique factorization domains* and *Euclidean domains*. Also as is customary, we will examine relationship between these various types of rings.

Then we will then spend some time discussing one of the most important rings, namely, the ring of polynomials over a field. This material will be critical to us when we study field theory and Galois theory in the next volume. Finally, we discuss certain finiteness conditions on rings, specifically, the *ascending chain condition* and the *descending chain condition* on ideals.

As you study the various branches of abstract algebra, you may find—as I think many do—that you like one branch of algebra more than another. However, it is important to keep in mind that there are many dependencies between the various branches and so despite these feelings, it is important to gain some measure of fluency in each branch of algebra to be successful in the other branches.

Index of Symbols

There is an index of symbols at the back of the book, in case you encounter a symbol that you do not recognize. Also, we will use the following symbols often:

1) $\mathbb{N} = \{0, 1, \ldots, \}$, the natural numbers, which *do* include 0,
2) $\mathbb{Z} =$ the integers,
3) $\mathbb{Z}^+ = \{1, 2, \ldots\}$, the positive integers,
4) $\mathbb{Q} =$ the rational numbers,
5) $\mathbb{R} =$ the real numbers,
6) $\mathbb{C} =$ the complex numbers.

Greek Alphabet

For reference, here is the Greek alphabet.

A α alpha	H η eta	N ν nu	T τ tau
B β beta	Θ θ theta	Ξ ξ xi	Υ υ upsilon
Γ γ gamma	I ι iota	O o omicron	Φ ϕ phi
Δ δ delta	K κ kappa	Π π pi	X χ chi
E ϵ epsilon	Λ λ lambda	P ρ rho	Ψ ψ psi
Z ζ zeta	M μ mu	Σ σ sigma	Ω ω omega

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