Abstract Algebra
A Comprehensive Introduction

Volume 2: Group Theory

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To Donna

Preface to the Series

This is a multi-volume series on abstract algebra, designed for the serious undergraduate or beginning graduate student. Dependencies among the first three volumes of this series, with hopefully more volumes to come, are as follows.

1) Volume 1: *Linear Algebra* can be read independently of the other two volumes.
2) Volume 2: *Group Theory* can be read independently of the other two volumes.
3) Volume 3: *Ring and Field Theory* does reference material on both linear algebra and group theory, but any reader with a modest acquaintance with these subjects should be able to read this volume without difficulty.
4) Volume 4: *Order and Lattices* can be read indepent=dently of theother volumes.

Thus, one has the option of starting this series with either the volume on group theory or the volume on linear algebra.

As the title suggests, this series is a *comprehensive introduction* to abstract algebra, where by *introduction*, I mean that the book starts more or less at the beginning, assuming no prior knowledge of abstract algebra. The only prerequisites for this series are an understanding of basic mathematical tools as found in a typical “transition course” and a solid understanding of elementary linear algebra as taught in a “relatively serious” lower division course. Certainly, some experience beyond these prerequisites will no doubt be of significant benefit in absorbing the material presented in this series.

Realizing that this may be a first exposure to rigorous abstract mathematics for some readers, I have tried to write the discussions and the proofs with an eye towards developing sound mathematical thinking patterns on the part of the reader. Proofs that appear in textbooks often do not show the *thought pattern* behind the discovery of the proof, often preferring elegance or conciseness over insight or motivation. However, showing a bit of the motivation behind a proof is in my opinion extremely valuable for students. Of course, one can easily get carried away with this idea, so I have tried to temper it with an eye towards not bloating these volumes or slowing down the flow of the books.

By *comprehensive*, I mean that the books in this series include a somewhat wider array of topics (some marked as optional or placed in topics chapters) than is often seen in elementary treatments, so that hopefully *all* readers will find something new and of interest in these books. My goal is for all readers to leave the series with a thorough grounding in the fundamentals of abstract algebra, well prepared to attack more advanced treatments if desired.

It seems that the current trend in mathematical education is to motivate abstract concepts by introducing *applications* as quickly as possible, in an effort to satisfy those students whose overriding question is: Of what use is this material?

While I certainly respect the views of those whose main concern is whether or not the subject matter at hand has applications to the “real” world, I have chosen to take a more abstract approach to the subject of abstract algebra in this series. I am a pure mathematician and appreciate mathematics as an *art form*, as well as the cornerstone of all science and technology. Merriam-Webster defines art as follows:

> something that is created with imagination and skill and that is beautiful or that expresses important ideas or feelings

What could possibly fit this description more accurately than mathematics and in particular, abstract algebra?

Preface To This Volume

This volume in the series is devoted to group theory. I recognize that instructors (including myself) often operate under time pressure that requires choices to be made in topic coverage. Accordingly, I have marked some of the topics in this book as *optional*, meaning that those topics can be skipped or assigned as independent reading without jeopardizing an understanding of subsequent material. Here is a list:

1) A brief and very general discussion of the classification problem for finite simple groups.
2) A brief discussion of universality as it relates to free groups.
3) An example showing that subgroups of a finitely-generated group need not be finitely generated.
4) A proof that the group $\mathbb{Z}_p^*$ is cyclic for p prime.
5) A discussion of the subgroup structure of the additive rationals.
6) A *very* brief discussion of Hamiltonian groups (nonabelian groups in which every subgroup is normal).
7) A discussion of conjugation in the alternating group as it relates to conjugation in the full symmetric group, specifically, which S_n-conjugacy classes are also A_n-conjugacy classes and which S_n-conjugacy classes split into (two) A_n-conjugacy classes.
8) An historical look at the way Evariste Galois viewed groups.

Organization of the Book

I believe it is somewhat customary at this point to summarize the topic coverage of the book.

Appendices

I should first mention that the book has two appendices. Appendix A contains some background information in preparation for studying the main text and Appendix B is a brief but very important introduction to partially ordered sets, lattices and intersection structures. The two appendices taken together amount to only about 15 pages, some or all of which may already be familiar to you, but *we will assume that you have familiarized yourself with the information in both appendices before reading the text itself*.

Chapter 1

The first chapter of the book is more about *context* than actual mathematics. Its aim is to give you an overview of the subject of abstract algebra and to examine the customary path that one takes in studying algebraic structures, such as groups or vector spaces, by describing several *common themes* encountered in the study of these structures. These common themes include the notions of substructure, quotient structure and structure-preserving map, for example. This discussion is phrased in the context of an arbitrary algebraic structure which I call a *widget*. If desired, the material on common themes can be skipped, but I believe (and have been told so by many of my students) that those who continue their study of abstract algebra beyond group theory will benefit from this overview.

Chapter 2

Chapter 2 begins group theory proper, starting with the definition and the basic properties of groups. We describe several examples of groups (quaternion, dihedral, general linear and so on) and take a close look at cyclic groups. We also discuss element orders and conjugacy and provide the basic ideas behind subgroups and group homomorphisms, which are studied in more detail later in the book. A brief but optional discussion of the remarkable classification problem for finite simple groups is also included. One goal of that discussion is to point out the vastness and depth of the subject and how much more there still is to discover about groups!

Chapter 3

Chapter 3 describes the two most important families of groups—the symmetric groups and the free groups. This discussion is placed relatively early in the book because these two families of groups are fundamental to an understanding of group theory and are also very useful in providing both examples and exercises. However, the early discussion of symmetric groups covers only the basic concepts, leaving other topics, such as the simplicity of A_n for $n \neq 4$ for a later chapter. The discussion of free groups comes earlier than in most other books at this level, but the approach is elementary and should be well within the grasp of the reader. One point that is emphasized is that the symmetric groups and the free groups hold opposite positions within the theory of groups—in the free groups, *nothing* is going on other than what is absolutely necessary in order to form a group, whereas in the symmetric groups, *everything* is going on, as Cayley pointed out in 1854.

Chapter 4

Despite the comments made earlier about the role (or rather the lack thereof) of applications in this series, I feel compelled to at least take a bit of time to describe very briefly and in very general terms how group theory is relevant other areas of mathematics, including geometry, analysis, algebraic topology, number theory and cryptology. The chapter can be skipped without jeopardizing an understanding of the material that follows.

Chapter 5

In Chapter 5, we take a closer look at subgroups, emphasizing the subgroup lattice of a group. Lagrange's theorem, generating sets and external direct products are also discussed in this chapter.

Chapter 6

It has always seemed to me that an inquiring student will be wondering at this point (or perhaps even earlier) how or why one actually *defines* a group that does not come in any obvious way from some well known mathematical concept, such as matrices or functions or by some physical concept, such as symmetry. In Chapter 6, I introduce the notion of a *group template* as a tool for identifying groups with desired properties. This chapter also contains a discussion of finitely-generated groups and whether subgroups of a finitely-generated group need be finitely generated.

Chapter 7

In Chapter 7, we take a close look at some well known families of groups other than the symmetric and free groups, which have already been discussed. These include the quaternion group, the Euclidean groups, the dihedral groups and the additive rationals.

Chapter 8

Chapter 8 is devoted to normal subgroups and quotient groups. There is also a prominent discussion of subgroups of order equal to half the order of the group, which I refer to for want of a better name as *gigantic subgroups*. (I would have preferred the term *monster subgroup* but monsters already exist in group theory!)

Chapter 9

Chapter 9 is devoted to internal direct products and sums, Cauchy's theorem and the classification of finite abelian groups. The history of Cauchy's theorem, or rather the proof thereof, is rather interesting and reminds us that we should never stop looking for more enlightening proofs of even the most basic results. The structure theorem for finite abelian groups is perhaps the most complex topic in this book—especially the matter of "breaking off" a cyclic subgroup from a p-primary component.

Chapter 10

Chapter 10 covers kernels, natural projections, the isomorphism theorems and the correspondence theorem, which can be succinctly described as a normality-preserving, index-preserving order isomorphism of certain subgroup lattices. While this description may seem a bit too much for an elementary course, I think that it provides not only the correct insight into the correspondence theorem, but also the best chance to *remember* the theorem.

Chapter 11

Chapter 11 continues and concludes our investigation of the symmetric groups, including the parity theorem, a short discussion of popular generating sets for S_n and A_n, the simplicity of A_n for $n \neq 4$ and the normal subgroups of S_n. We also examine the question of the relationship between conjugacy in S_n and conjugacy in A_n, although this topic is optional.

Chapter 12

Chapter 12 is devoted to group actions. Three such actions are emphasized: Translation by a group G on a quotient group G/H, conjugation by G on the conjugates of a subgroup of G and conjugation by G on G itself. There is also a brief discussion of transitive group actions.

Chapter 13

Chapter 13 describes the Sylow theorems. The approach to the Sylow theorems begins by assuming that a group G has a *normal* Sylow subgroup S. Much can be learned relatively easily under this assumption. In particular, it is easy to see that *every* p-subgroup H of G must be contained in S because HS is a p-subgroup that contains S. This opens the door to the general Sylow theory, as it were.

Of course, normal Sylow subgroups are somewhat rare, but every Sylow subgroup is normal in its *normalizer* and so the headway made under the assumption that S is normal can be translated from the setting of the group G to the normalizer $N_G(S)$. This approach seems to ease the road to the full Sylow theorems—All that is necessary at this point is to find and explore the "right" group action.

We next catalog some approaches to finding normal subgroups of a group, to wit,

1) determining by a simple arithmetic technique I call the *p-condition* that $n_p = 1$, where n_p is the number of Sylow p-subgroups of G,
2) counting the elements of order p to show that there is room for only one Sylow p-subgroup,
3) examining the kernel of a suitable group action,
4) finding a subgroup H of G whose index is equal to the smallest prime divisor of G.

We remark in the text that a little computer programming shows that among the orders up to 10000 (not including prime powers) there are only 569 orders (less than 6% of the total) that are *not* amenable to the p-condition! Thus, for the vast majority of orders up to 10000, groups have normal Sylow p-subgroups for *some* prime p. So we see that even this relatively simple criterion shows that there are relatively few simple groups, at least among groups of relatively small order.

Next, we use the not inconsiderable ammunition that we have gathered to take a detailed look at groups of order pq, where $p < q$ are primes.

Index of Symbols

There is an index of symbols at the back of the book, in case you encounter a symbol that you do not recognize.

The Greek Alphabet

It seems that mathematicians never have enough available symbols. In particular, the usual Roman alphabet does not supply enough symbols to denote variables of different types. Accordingly, mathematicians find it necessary to reach out to other alphabet systems. It is fair to say that all mathematicians (and most mathematics books) make considerable use of the Greek alphabet, shown in the table below. If you intend to study mathematics seriously, some familiarity with this alphabet is essential (with the possible exception of omicron and upsilon!).

A α alpha	H η eta	N ν nu	T τ tau
B β beta	Θ θ theta	Ξ ξ xi	Υ υ upsilon
Γ γ gamma	I ι iota	O o omicron	Φ ϕ phi
Δ δ delta	K κ kappa	Π π pi	X χ chi
E ϵ epsilon	Λ λ lambda	P ρ rho	Ψ ψ psi
Z ζ zeta	M μ mu	Σ σ sigma	Ω ω omega

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