Abstract Algebra

A Comprehensive Introduction

Volume 4: Field and Galois Theory
Version 2.0.0

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Preface to the Series

This is a multi-volume series on abstract algebra, designed for the serious undergraduate or beginning graduate student. Dependencies among the volumes in this series are as follows.

1) Volume 1: *Linear Algebra* can be read independently of the other volumes.
2) Volume 2: *Group Theory* can be read independently of the other volumes.
3) Volume 3: *Ring Theory* does reference material on both linear algebra and group theory, but any reader with a modest acquaintance with these subjects should be able to read this volume without difficulty.
4) Volume 4: *Field Theory* does reference material on linear algebra, group theory and ring theory, but any reader with a modest acquaintance with these subjects should be able to read this volume without difficulty.
5) Volume 5: *Order and Lattices* can be read independently of the other volumes.

As the title suggests, this series is a *comprehensive introduction* to abstract algebra, where by *introduction*, I mean that the book starts more or less at the beginning, assuming no prior knowledge of abstract algebra. The only prerequisites for this series are an understanding of basic mathematical tools as found in a typical "transition course" and a solid understanding of elementary linear algebra as taught in a "relatively serious" lower division course. Certainly, some experience beyond these prerequisites will no doubt be of significant benefit in absorbing the material presented in this series.

By *comprehensive*, I mean that the books in this series include a somewhat wider array of topics (some marked as optional or placed in topics chapters) than is often seen in elementary treatments, so that hopefully *all* readers will find something new and of interest in these books. My goal is for all readers to leave the series with a thorough grounding in the fundamentals of abstract algebra, well prepared to attack more advanced treatments if desired.

It seems that the current trend in mathematical education is to motivate abstract concepts by introducing *applications* as quickly as possible, in an effort to satisfy those students whose overriding question is: Of what use is this material?

While I certainly respect the views of those whose main concern is whether or not the subject matter at hand has applications to the "real" world, I have chosen to take a more abstract approach to the subject of abstract algebra in this series. I am a pure mathematician and appreciate mathematics as an *art form*, as well as the cornerstone of all science and technology. Merriam-Webster defines art as follows:

> something that is created with imagination and skill and that is beautiful or that expresses important ideas or feelings

What could possibly fit this description more accurately than mathematics and in particular, abstract algebra?

Preface to this Volume

This volume of the series is devoted to the elementary theory of fields. For this volume, readers are expected to have a grasp of elementary linear algebra, some elementary ring theory and some elementary group theory.

I have assumed that for many readers, this series is their first experience with a serious course in abstract mathematics, having perhaps had only calculus, discrete mathematics, elementary differential equations and the aforementioned elementary linear algebra prior to undertaking this course. For that reason, I tried in the first volume to engage the student as much as possible by moving rather casually at times. But the honeymoon is in some sense over and in the interest of conciseness, I will be a little more formal in this volume. Actually, I regard this as a service to the reader, because as he or she progresses into abstract mathematics, formal exposition will be the overwhelming rule, rather than the exception, and there is no time like the present to begin to get used to this situation. In writing the first volume, I constantly asked myself if the current discussion was sufficiently motivated. However, I will not do this as much in this volume.

Field Theory Has a Different Feel

Although the theories of vector spaces, groups and rings have many commonalities, as you embark on a study of elementary field theory, you may notice that this subject has fewer themes that are common to the other subjects. For example, although there are quotient spaces, quotient groups and quotient rings, there is no such thing as a quotient *field*. Nor is the cartesian product of two fields a field, as is the case for vector spaces, groups and rings. Also, all homomorphisms of fields are injective and so have trivial kernel, which contributes to the differences between field theory and the other branches of abstract algebra. As you may imagine, this is due to one defining property of fields, namely, that all *nonzero* elements must have a multiplicative inverse. Simply put, the fact that the additive identity has no multiplicative inverse has profound effects on the nature of fields.

Index of Symbols

There is an index of symbols at the back of the book, in case you encounter a symbol that you do not recognize. Also, we will use the following symbols often:

1) $\mathbb{N} = \{0, 1, \ldots, \}$, the natural numbers, which *do* include 0,
2) $\mathbb{Z} =$ the integers,
3) $\mathbb{Z}^+ = \{1, 2, \ldots\}$, the positive integers,
4) $\mathbb{Q} =$ the rational numbers,
5) $\mathbb{R} =$ the real numbers,
6) $\mathbb{C} =$ the complex numbers.

Greek Alphabet

It seems that mathematicians never have enough symbols. In particular, the usual Roman alphabet does not supply enough symbols to denote variables of different types. Accordingly, mathematicians find it necessary to reach out to other alphabet systems.

Some of the older classic abstract algebra textbooks (notably by Nathan Jacobson) employ the **Fraktur** alphabet shown below.

$\mathfrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$
$\mathfrak{abcdefghijklmnopqrstuvwxyz}$

However, as much as the reader might enjoy the added confusion that this alphabet provides while trying to learn algebra, we will not use it in this book. (For example, compare the upper case A ($\mathfrak{A}$) with the upper case U ($\mathfrak{U}$).)

It is fair to say that all mathematicians (and most mathematics books) make extensive use of the Greek alphabet, shown in the table below. If you intend to study mathematics seriously, knowledge of this alphabet is essential.

A α alpha	H η eta	N ν nu	T τ tau
B β beta	Θ θ theta	Ξ ξ xi	Υ υ upsilon
Γ γ gamma	I ι iota	O o omicron	Φ ϕ phi
Δ δ delta	K κ kappa	Π π pi	X χ chi
E ϵ epsilon	Λ λ lambda	P ρ rho	Ψ ψ psi
Z ζ zeta	M μ mu	Σ σ sigma	Ω ω omega

Good luck and thanks for reading.

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