Transition to Advanced Mathematics

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Preface

The purpose of this book/course is to prepare you for the study of advanced mathematics, specifically upper division university-level mathematics. This is actually a three-part process.

1) The first part of the process involves the study of the rudiments of *mathematical logic*, which will provide some training in how to *think logically*, a skill that is absolutely critical to the ability to learn advanced mathematics.
2) The second part of the process is a careful look at *proof techniques*, which is essentially an application of logical reasoning to mathematics.
3) Finally, there are several topics in mathematics that are essential to the study of advanced mathematics and yet seem to find no home in any traditional math class, so this is the perfect place to examine these topics.

Courses such as elementary calculus, elementary linear algebra and elementary differential equations, as generally taught in the first two years of undergraduate study are not representative of mathematics in general, for they tend to be recipe driven, that is, the main goal of these elementary classes is to teach you how to solve problems by *following a script*. Emphasis is on the *how* rather than the *why* and there is in general very little discussion of the underlying theory behind the subject.

All this changes with upper-division courses. Emphasis switches to an understanding of the underlying theory and to the construction of proofs to justify mathematical statements. This book provides the preparation necessary to meet the challenges of upper-division mathematics courses.

Prerequisites

Prerequisites for this course are minimal. The most important prerequisite is a *strong interest* in mathematics. Of course, any lower-division classes in mathematics will help by providing some experience, but they are not strictly speaking necessary. There are a few references to elementary calculus, mostly in examples, but the reader can simply skip these references if necessary.

While on the subject of prerequisites, it is difficult to discuss mathematics without the need to mention *functions* from time to time. We will devote an entire chapter in this book to a detailed look at functions, but before reaching that chapter, we will need to use the concept now and then. Accordingly, we have included a very brief discussion of functions at the end of this Preface. It may very well be that all of this material is familiar to you, but a quick glance can do no harm.

Exercises

For the most part, the exercises are sprinkled throughout the text in the hope that it will encourage you to attempt the exercises in real time, as it were. One notable exception is the chapter on proof techniques because part of learning proof techniques is to determine which technique should be used and I didn't want to give away any hints by the placement of the exercises!

A Brief Look at Functions

We will devote an entire chapter of this book to a detailed look at the important concept of a function, but prior to that chapter, we will have a few occasions to discuss functions (such the successor function and indexing functions). Accordingly, we should take a very brief look at the definition of a function and some of their basic properties now. This is material with which you may already be familiar, but at least it will set the terminology so we are all on the same page, as they say.

Actually, there are two somewhat different definitions of a function and we will discuss this in some detail in a chapter devoted to functions. Until we reach that chapter, we can adopt the following version of the definition.

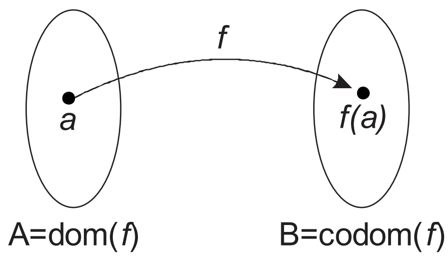

Figure P.1: A function

Definition *Let A and B be sets. As shown in Figure P.1, a* **function** *f consists of three things:*
1) A nonempty set A, called the **domain** *of the function f and denoted by $\operatorname{dom}(f)$.*
2) A nonempty set B, called the **codomain** *of the function f and denoted by $\operatorname{codom}(f)$.*
3) An assignment of an element of B to each element of A.
In this case, we say that f is a function **from** *A* **to** *B and write*

$$f: A \to B$$

Functions are also sometimes called **maps**. *We write $b = f(a)$ to denote the fact that the function f maps the element $a \in A$ to the element $b \in B$. In this case, $f(a)$ is called the* **image** *of a under f and a is called a* **preimage** *of b under f. If $f: A \to A$ is a function form A to itself, we say that f is a function* **on** *A.*□

We can also define the image or range of a function.

Definition *Let $f: A \to B$ be a function. The* **image** *of f, denoted by $\operatorname{im}(f)$ and also called the* **range** *of f, denoted by $\operatorname{ran}(f)$, is the set of all elements of the codomain B that are assigned to some element of the domain A by f, that is, $\operatorname{im}(f)$ is the set of images of all of the elements in the domain of f, in symbols*

$$\operatorname{im}(f) = \operatorname{ran}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

Note that $\operatorname{im}(f) \subseteq \operatorname{codom}(f)$.□

Injective Functions

It is possible that two different elements in the domain of a function can be sent to the *same* element in the codomain, as illustrated in Figure P.2. Put another way, an element of B may have more than one preimage. Functions that do *not* have this property are called *injective functions*, *injections* or *one-to-one functions*.

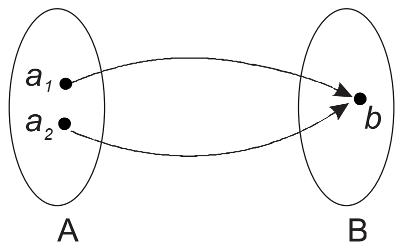

Figure P.2

Definition *Let $f: A \to B$ be a function. If no two distinct elements in A are assigned to the same element in B, then f is said to be* **injective** *or an* **injection** *or* **one-to-one**. *In symbols, a function is injective if*

$$a_1 \neq a_2 \quad \Rightarrow \quad f(a_1) \neq f(a_2)$$

or equivalently, if

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2$$

□

Surjective Functions

We have said that the image of a function is a subset of its codomain. Functions for which the image and codomain are equal have a special name.

Definition *If a function $f: A \to B$ has the property that*

$$\operatorname{im}(f) = \operatorname{codom}(f)$$

then the function f is said to be **surjective** *or a* **surjection**. *We also say that f maps A* **onto** *B.*□

Bijective Functions

A function $f: A \to B$ that is both injective and surjective is **bijective**, also called a **bijection**. The bijections are precisely the functions that have inverses, that is, functions $f^{-1}: B \to A$ with the property that

$$f^{-1} \circ f = \iota_A \quad \text{and} \quad f \circ f^{-1} = \iota_B$$

where $\iota_A: A \to A$ is the identity function on A and similarly, ι_B is the identity function on B.

The Greek Alphabet

From time to time, we will use a few Greek letters. For reference, here is a table of the Greek alphabet.

A α alpha	H η eta	N ν nu	T τ tau
B β beta	Θ θ theta	Ξ ξ xi	Υ υ upsilon
Γ γ gamma	I ι iota	O o omicron	Φ ϕ phi
Δ δ delta	K κ kappa	Π π pi	X χ chi
E ϵ epsilon	Λ λ lambda	P ρ rho	Ψ ψ psi
Z ζ zeta	M μ mu	Σ σ sigma	Ω ω omega

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